

Analysis of Critical Points for Nonconvex Optimization

by

Alp Simsek

Submitted to the MIT Department of Electrical Engineering and
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Abstract

In this thesis, we establish sufficient conditions under which an optimization problem has a unique local optimum. Motivated by the practical need for establishing the uniqueness of the optimum in an optimization problem in fields such as global optimization, equilibrium analysis, and efficient algorithm design, we provide sufficient conditions that are not merely theoretical characterizations of uniqueness, but rather, given an optimization problem, can be checked algebraically. In our analysis we use results from two major mathematical disciplines. Using the mountain pass theory of variational analysis, we are able to establish the uniqueness of the local optimum for problems in which every stationary point of the objective function is a strict local minimum and the function satisfies certain boundary conditions on the constraint region. Using the index theory of differential topology, we are able to establish the uniqueness of the local optimum for problems in which every generalized stationary point (Karush-Kuhn-Tucker point) of the objective function is a strict local minimum and the function satisfies some non-degeneracy assumptions. The uniqueness results we establish using the mountain pass theory and the topological index theory are comparable but not identical. Our results from the mountain pass analysis require the function to satisfy less strict structural assumptions such as weaker differentiability requirements, but more strict boundary conditions. In contrast, our results from the index theory require strong differentiability and non-degeneracy assumptions on the function, but treat the boundary and interior stationary points uniformly to assert the uniqueness of the optimum under weaker boundary conditions.

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Chapter 1

Introduction

The seemingly theoretical question of demonstrating that a function has a unique local optimum has significant implications in various different contexts in optimization theory, equilibrium analysis, and algorithm design. Uniqueness of a local optimum plays a central role in the design of numerical algorithms for computing the global optimal solution of an optimization problem. When the optimization problem has multiple local optima, gradient-based optimization methods do not necessarily converge to the global optimum, since these methods, unaware of the global structure, are guided downhill by the local structure in the problem and can get attracted to a globally sub-optimal local optimum. It is conceivable that theoretical uniqueness results can be programmed in an optimization software to develop uniqueness aware algorithms that switch methods depending on the multiplicity of optima and thus are more efficient. Subsequently, the problem of establishing sufficient conditions under which a function has a unique local optimum (hereafter the uniqueness problem) is not merely a mathematical curiosity but is of practical importance in global optimization.

The uniqueness problem has been prominent in equilibrium analysis in a number of aspects. First, in some game theoretic models, asserting the uniqueness of the optimum in an optimization problem is sufficient to establish the existence of an equilibrium. Consider a setting with multiple agents with conflicting interests and assume that we are interested in studying the appropriate equilibrium concept (Nash,

Wardrop, etc. ¹). Consider the optimization problem of each individual in computing his best response as a function of the strategies of other players (for the case of a Nash equilibrium) and as a function of the endogenous control parameter (price for a market or congestion level for a transport/communication network) which will in turn be a function of the individual agents' actions (for the case of a competitive or Wardrop equilibrium). If the structure of the individual's optimization problem is such that the problem admits a unique local optimum ², then the first order optimality conditions can be used as an equivalent characterization of the global optimum. This characterization can in turn be used to assert the existence of an equilibrium under continuity assumptions (see Osborne-Rubinstein [40]).

Second, the equivalent characterization of equilibrium by the first order optimality conditions of the optimization problem can be used to investigate the equilibrium properties. One area in which this approach has been used extensively is the emerging paradigm of resource allocation among heterogeneous users in a communication network. Consider a communication model in which a network consisting of links with load dependent latency functions is shared by noncooperative users who selfishly route traffic on minimum latency paths. The quality of a routing in this setting is measured by the total latency experienced by all users. When the utility functions of the users and the latency functions of the links satisfy certain convexity and continuity properties, it can be shown that each individual faces an optimization problem which admits a unique solution. Then, the routing of each user can be characterized by the first order optimality conditions, which in turn can be used to measure the quality of the routing in equilibrium. Consequently, the quality of the routing obtained in the noncooperative setting can be compared with the quality of the minimum possible latency routing, which would be obtained if a central planner with full information and full control coordinated the actions of each user. The difference in quality between the two cases measures the degradation in network performance caused by the

¹See Haurie-Marcotte [27] for a definition of these equilibria.

²It is also essential to assert the existence of a global optimum, which does not necessarily follow from the uniqueness of the local optimum. In this thesis, we are interested in studying the uniqueness property. We thus assume that a global solution exists.

selfish behavior of noncooperative network users and has been the subject of much recent study (see, for example, Roughgarden [43, 44], Roughgarden-Tardos [45, 46], Acemoglu-Ozdaglar [2, 3]).

Third, the uniqueness properties of an optimization problem imply the uniqueness of the equilibrium in certain two-level Stackelberg games, in which one player (the leader) chooses an action first and the remaining players (the followers) observe the leader's action afterwards (see Osborne-Rubinstein [40], Vega-Redondo [60]). As an example, consider a communication model by Acemoglu-Ozdaglar [39] in which, links are owned and priced by a profit-maximizing monopolist (Stackelberg leader), and noncooperative users (Stackelberg followers) selfishly decide on the amount and routing of the traffic they send taking into account the price and the latency on the links. Characterizing the optimal actions of the players correspond to finding *the Subgame Perfect Equilibrium (SPE)* (see Osborne-Rubinstein [40] or Vega-Redondo [60]) of the two-level game. For the purposes of the resource allocation problem, the SPE can be associated with the optimal solution of the profit maximization problem faced by the monopolist, hence if the optimization problem admits a unique solution then the model has a unique equilibrium. However, the profit function of the monopolist (and in general, the Stackelberg leader) is nonconvex due to the two-level structure of the problem, and consequently, it is not possible to establish the uniqueness of the optimum using convexity.

Beyond the static equilibrium analysis setting, the uniqueness problem has also implications on the design of efficient distributed algorithms for resource allocation in communication networks. If the allocation problem can be formulated as a convex optimization problem, then the traditional network optimization techniques can be used to devise distributed algorithms that are guaranteed to converge to the unique optimum which corresponds to the equilibrium of the model (see Kelly et al. [31], Low-Lapsley [33]). For the case when the problem is nonconvex but has a unique local optimum, it may be possible to develop distributed algorithms with provable convergence behavior. In recent work, Huang-Berry-Honig [24] develop such an algorithm for optimal power control in wireless networks, which is a nonconvex problem

due to the physical nature of the wireless interference. They show the uniqueness of the optimum for the nonconvex optimization problem, which plays a key role in their algorithm and convergence results.

The uniqueness of the local optimum is straightforward to establish when the objective function satisfies convexity properties, but the nonconvex case is not completely understood. The traditional optimization results assert the uniqueness of the local optimum when the objective function is (strictly) quasi-convex (Bertsekas [11], Bertsekas-Nedic-Ozdaglar [12], Rockafellar [50], Rockafellar-Wets [51]). It can be shown that the convexity requirements of these results are not necessary conditions for the local optimum to be unique [cf. Section 2.1]. The literature on uniqueness for the nonconvex case can be divided into two classes. The first set of results provide (partially) equivalent necessary and sufficient conditions for the optimization problem to have a unique optimum (Chapter 6 of Avriel [6], Section 4.2 of Ortega-Rheinboldt [40], Chapter 9 of Avriel et al. [5]). However, the equivalent conditions established by these results are theoretical characterizations which, given an optimization problem, cannot be checked algebraically. The second set of results provide sufficient conditions which can be checked algebraically but which apply to specific types of problems. These results either consider problems which can be transformed into the well understood convex case (Chapter 8 of Avriel et al. [5]) or they consider special families of problems in which the objective function is quadratic, is a product, a ratio, or a composite of convex (concave) functions, or has a similar well-specified structure (Chapters 5 and 6 of Avriel et al. [5]).

For establishing the uniqueness of the optimum in nonconvex problems, the contrast between equivalent theoretical characterizations and sufficient conditions that can be checked algebraically is well described by the following passage by Avriel et al. [5].

Unfortunately, the standard definitions are not often easily applicable to decide whether a given function is generalized concave or not. Even for functions in two variables it may be quite difficult to verify the defining inequalities of quasiconcavity or pseudoconcavity, for example. By restricting ourselves to specific classes of functions, having

a certain algebraic structure, we can hope to find more practical characterizations.

In this thesis, our main contribution is to present sufficient conditions which can be checked algebraically and which assert the uniqueness of the local optimum for general nonconvex optimization problems, without imposing a special structure on the problem. The following condition plays a central role in our analysis.

(Q1) Every stationary point of the objective function is a strict local minimum.

Note that this condition can be checked algebraically to assert the uniqueness of the local optimum. In particular, assuming that the objective function is twice continuously differentiable, the stationary points for a minimization problem are characterized by the first order optimality conditions, and every stationary point that satisfies the second order optimality conditions is a strict local minimum. Then, given the minimization problem, to check condition (Q1), one needs to check if the second order conditions are satisfied at points which satisfy the first order conditions. In terms of feasibility, checking (Q1) is easier than widely used checks for convexity in which one checks if the second order conditions (positive definiteness of the hessian) are satisfied at every point. Furthermore, this condition is evidently easier to establish than the extensively studied theoretical characterizations of generalized convexity, an example of which is the following condition.

(D) Given any pair of points in the function domain, the points are connected by a path within the domain such that the value of the objective function on the path is not greater than the maximum function value of the end-points.

We investigate two different disciplines to establish conditions similar to (Q1) which can be checked algebraically and which are sufficient for the uniqueness of the local optimum in an optimization problem. In Chapter 2, we investigate the mountain pass theory which is well suited to analyzing the relationship between (Q1) and the uniqueness problem. When the objective function is defined on an unconstrained region, we show that if the function satisfies (Q1) and a compactness-like condition,

then it has at most one local minimum. When the objective function is defined on a constrained region, we show that if the function satisfies (Q1) and additional conditions on the boundary of the constraint region, then it has at most one local minimum. In Chapter 3, we investigate the implications of differential topology, in particular, the topological index theory, on the uniqueness problem. Using the Poincare-Hopf Theorem, when the objective function satisfies certain conditions on the boundary of the constraint region, we establish a relationship between the local properties of the function at its stationary points and the topology of the underlying region. We then observe that if the function satisfies (Q1), then it cannot have two local minima or else this relationship would be violated. As in Chapter 2, this allows us to establish the uniqueness of the optimum when the function satisfies (Q1) and certain boundary conditions. We then present our extension of the Poincare-Hopf Theorem, which holds for more general regions, generalizes the notion of a critical point to include the boundary critical points, and dispenses away with the boundary condition of the original theorem. Using our generalized Poincare-Hopf theorem, we establish a relationship between the local properties of the objective function at its generalized stationary points (Karush-Kuhn-Tucker points) and the topology of the underlying constraint region. We then use this relationship to show uniqueness results which replace (Q1) with an appropriate generalization for the boundary stationary points and which do not need boundary conditions.

In this thesis, we show that the constrained mountain pass theory and the topological index theory, which are two very different disciplines, provide similar conclusions regarding the uniqueness problem. Nevertheless, the assumptions used reflect the nature of each field and thus are not identical. The uniqueness results we establish using the constrained mountain pass theory require the objective function to be continuously differentiable, whereas the ones we establish using the index theory require it to be twice continuously differentiable. Further, the index theory arguments require non-degeneracy assumptions on the function, which, in the optimization framework, boils down to the strict complementary slackness condition (see Bertsekas [11]). In a tradeoff for the additional structural assumptions required by the index theory,

we establish stronger uniqueness results which treat the boundary and the interior stationary points uniformly. In essence, a boundary condition is needed in our uniqueness results from the constrained mountain pass theory, since the arguments do not account for the boundary stationary points and thus cannot allow them. Our uniqueness results from the topological index theory unify the two requirements [(Q1) and the boundary condition] for uniqueness by treating the boundary and the interior points uniformly.

Finally, we note that the sufficiency results we establish for the uniqueness of the local optimum have applications in practical nonconvex problems. Our results from the constrained mountain pass theory can be used to establish the uniqueness of equilibrium in the Acemoglu-Ozdaglar [1] network pricing model. Our results from the topological index theory can be used to give an alternative proof for the uniqueness of the optimum in the wireless interference model of Huang-Berry-Honig [24]. Huang-Berry-Honig [24] proves uniqueness using the special (convex-transformable) structure of the optimization problem. In contrast, our results can be used to establish uniqueness in a more general setting of wireless interference problems including the particular convex-transformable problem analyzed in [24].

Related Work

The (unconstrained) Mountain Pass Theorem (Theorem 3.1) we present in Chapter 3 was first developed by Ambrosetti-Rabinowitz [4]. The first part of the constrained mountain pass theorem (part (i) of Theorem 3.3) we present can be derived from a constrained mountain pass result by Tintarev [59]. Part (ii) of the same theorem appears to be a new result. Also, to our knowledge, the investigation of these theorems in an optimization framework to provide sufficient conditions for the uniqueness of the local optimum is new.

The original Poincare-Hopf Theorem we present in Chapter 4 is a well-known theorem in differential topology and dates back to Henri Poincare (1854-1912). Morse [36], Gottlieb [21], and Gottlieb-Samaranayake [22] prove generalized Poincare-Hopf theorems on smooth manifolds with boundary which extend the notion of critical

points to include the boundary critical points. Our generalization (Theorem 4.1) which further extends these generalized theorems to regions that are not necessarily smooth manifolds appears to be new. The extension idea used in the proof of the theorem is originally due to Morse [36].

Our application of the Poincare-Hopf Theorem on an optimization problem to establish sufficient conditions for the uniqueness of the local optimum appears to be new, yet there are a number of applications of the topological index theory in establishing the existence and the uniqueness of zeros of vector valued functions and correspondences. Dierker [20], Mas-Colell [34], Varian [61], Hildenbrand-Kirman [25] use the Poincare-Hopf Theorem to prove the uniqueness of general equilibrium in an exchange economy. Ben-El-Mechaiekh et al. [9], and Cornot-Czarnecki [15, 16, 17] use versions of topological index theory to prove existence of fixed points for vector valued correspondences.

In [29], Jongen et al. prove results along the lines of Proposition 4.9 by generalizing Morse Theory to constrained regions, which is different than the generalized index theory approach we take. The application of Proposition 4.9 to obtain sufficient conditions for the uniqueness of the local optimum appears to be new.

Organization

The organization of this thesis is as following. In Chapter 2, we present convex and nonconvex optimization results which provide equivalent characterizations for the local optimum to be unique, but which cannot be algebraically checked. In Chapter 3, we introduce the mountain pass theory and derive conditions which assert the uniqueness of the local optimum. In Chapter 4, we introduce differential topology tools and derive sufficient conditions for uniqueness. In Chapter 5, we discuss future research directions.

Chapter 2

Optimization Theory and the Uniqueness Property

Notation and Preliminaries

All vectors are viewed as column vectors, and $x^T y$ denotes the inner product of the vectors x and y . We denote the 2-norm as $\|x\| = (x^T x)^{1/2}$. For a given set X , we use $\text{conv}(X)$ to denote the convex hull of X . Given $x \in \mathbb{R}^n$ and $\delta > 0$, $B(x, \delta)$ denotes the open ball with radius δ centered at x , i.e.

$$B(x, \delta) = \{y \in \mathbb{R}^n \mid \|y - x\| < \delta\}.$$

Given $M \subset \mathbb{R}^n$, we define the interior, closure, and boundary of M as

$$\text{int}(M) = \{x \in M \mid B(x, \delta) \subset M \text{ for some } \delta > 0, \}$$

$$\text{cl}(M) = \mathbb{R}^n - \text{int}(\mathbb{R}^n - M)$$

$$\text{bd}(M) = \text{cl}(M) - \text{int}(M).$$

For a given function $f : A \mapsto B$, $f|_C : C \mapsto B$ denotes the restriction of f to $C \subset A$, $f(C) \subset B$ denotes the *image of C under f* . For $D \subset B$ we denote the *pre-image of D under f* as

$$f^{-1}(D) = \{x \in A \mid f(x) \in D\}.$$

If f is differentiable at x , then $\nabla f(x)$ denotes the gradient of f at x . If f is twice differentiable at x , then $H_f(x)$ denotes the Hessian of f at x . If $f : A \times B \mapsto C$ and $f(., y) : A \mapsto C$ is differentiable at $x \in A$, then $\nabla_x f(x, y)$ denotes the gradient of $f(., y)$ at $x \in A$.

2.1 Convex Optimization and the Uniqueness Property

Consider a set $M \subset \mathbb{R}^n$ and a function $f : M \mapsto \mathbb{R}$. $x^* \in M$ is a *local minimum* of f over the set M if there exists $\epsilon > 0$ such that

$$f(x^*) \leq f(x), \quad \forall x \in M \text{ with } \|x - x^*\| < \epsilon. \quad (2.1)$$

$x^* \in M$ is a *global minimum* of f over M if

$$f(x^*) \leq f(x), \quad \forall x \in M. \quad (2.2)$$

x^* is called a *strict local minimum* [respectively, *strict global minimum*] if the inequality in (2.1) [respectively, (2.2)] is strict.

It is clear that every global minimum of f is a local minimum but not vice versa. We define a number of properties regarding local and global minima of f over M .

(P1) Every local minimum of f over M is a global minimum.

(P2) f has at most one local minimum over M .

(P3) (existence) f has a global minimum over M .

(P4) (uniqueness) f has a unique local minimum over M which is also the global minimum.

The goal of this thesis is to assess conditions under which (f, M) satisfies (P4), which we call the uniqueness property. It is clear that (f, M) satisfies (P4) if and only if (f, M) satisfies (P2) and (P3). The existence property (P3) is not studied in this

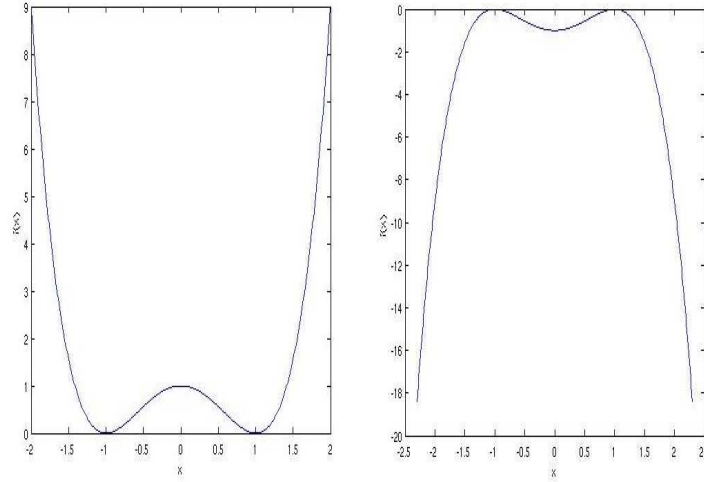


Figure 2.1: The function on the left satisfies (P1) but not (P2). The function on the right satisfies (P2) but not (P1).

work and will usually be assumed to hold in the analysis (usually by assuming that M is compact), thereby making (P2) the focus of the thesis. The following examples show that neither (P1) implies (P2), nor (P2) implies (P1).

Example 2.1 Let $M = [-2, 2]$ and

$$f(x) = (x^2 - 1)^2.$$

Then, (f, M) satisfies (P1) but not (P2), as seen from Figure 2.1.

Example 2.2 Let $M = \mathbb{R}$ and

$$f(x) = -(x^2 - 1)^2.$$

This time, (f, M) satisfies (P2) but not (P1) as seen from Figure 2.1.

However, we note that (P3) and (P2) together imply (P1). Hence given (P3) (possibly through compactness of M), (P2) is a stronger property than (P1). Figure 2.1 summarizes the relationship between these properties.

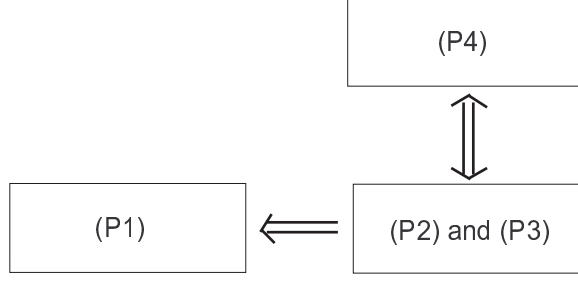


Figure 2.2: The relationship between properties regarding local and global minima.

We recall the notion of convexity for sets and functions. A set $M \subset \mathbb{R}^n$ is a *convex set* if

$$\alpha x + (1 - \alpha)y \in M, \quad \forall x, y \in M, \forall \alpha \in [0, 1].$$

Given a convex set $M \subset \mathbb{R}^n$, a function $f : M \mapsto \mathbb{R}$ is called *convex* if

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y), \quad \forall x, y \in M, \forall \alpha \in [0, 1]. \quad (2.3)$$

The function f is called *concave* if $-f$ is convex. The function f is called *strictly convex* if the inequality in (2.3) is strict for all $x, y \in M$ with $x \neq y$, and all $\alpha \in (0, 1)$.

The following proposition states the well-known result that convexity implies property (P2) (see Section 2.1 of Bertsekas-Nedic-Ozdaglar [12] for the proof).

Proposition 2.1 If $M \subset \mathbb{R}^n$ is a convex set and $f : M \mapsto \mathbb{R}$ is a convex function, then (f, M) satisfies (P1). If in addition f is strictly convex over M , then f also satisfies (P2).

We next recall the notion of a quasi-convex function, which is a generalization of convexity. Given a convex set $M \subset \mathbb{R}^n$, a function $f : M \mapsto \mathbb{R}$ is called *quasi-convex* if for each $x, y \in M$, we have

$$f(\alpha x + (1 - \alpha)y) \leq \max(f(x), f(y)), \quad \forall \alpha \in [0, 1]. \quad (2.4)$$

The function f is called *quasi-concave* if $-f$ is *quasi-convex*. The function f is called

*strictly quasi-convex*¹ if the inequality in (2.4) is strict for all $x \neq y$ and $\alpha \in (0, 1)$.

Quasi-convexity is closely related with the concept of a level set. Given $M \subset \mathbb{R}^n$, $f : M \mapsto \mathbb{R}$, and $\alpha \in \mathbb{R}$, we define the α *lower level set* of f as

$$L_\alpha = \{x \in M \mid f(x) \leq \alpha\}.$$

It can be shown that f is quasi-convex if and only if all non-empty lower level sets of f are convex².

It is well-known that quasi-convexity is a weaker notion than convexity, i.e., every convex [respectively, strictly convex] function is quasi-convex [respectively, strictly quasi-convex] but not vice-versa. The following is a generalization of Proposition 2.1 for quasi-convex functions (see Section 3 of Avriel et al. [5] or Section 4.5 of Bazaara-Shetty [13] for the proof).

Proposition 2.2 If $M \subset \mathbb{R}^n$ is a convex set and $f : \mathbb{R}^n \mapsto \mathbb{R}$ is a strictly quasi-convex function, then (f, M) satisfies (P1) and (P2).

Propositions 2.1 and 2.2 motivate us to consider conditions under which (f, M) satisfies (P1) and (P2). It is clear that quasi-convexity is not a necessary condition for (P1) and (P2) to hold. The next example illustrates this fact.

Example 2.3 Let $M = \mathbb{R}^2$ and $f : M \mapsto \mathbb{R}$, $f(x, y) = -e^{-x^2} + y^2$. Clearly this function has a unique local and global minimum at $(x, y) = 0$, and thus satisfies (P1) and (P2). However, it is neither convex nor quasi-convex as can be seen from Figure 2.3.

In this thesis, we investigate sufficient conditions for (P1) and (P2) which are weaker than convexity and which can be algebraically checked. For completeness, in the following section, we review results from nonconvex optimization literature that establish necessary and sufficient conditions for (P1) and, under additional assumptions, for (P2). We note, however, that these results provide theoretical character-

¹In optimization literature, there are various definitions of strict quasi-convexity with slight differences. We adopt the definition in Avriel et al. [5] and Ortega-Rheinboldt [41].

²See Chapter 4 of Ortega-Rheinboldt [41] or Chapter 3 of Avriel et al. [5] for the proof.

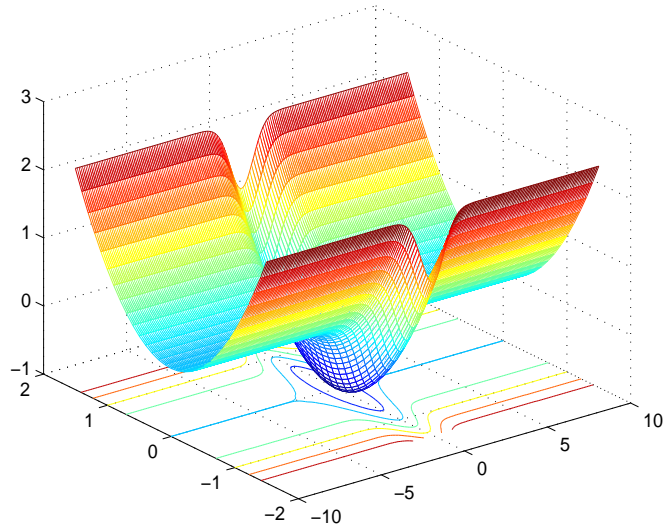


Figure 2.3: The function $-e^{-x^2} + y^2$ is not quasi-convex, as can be seen from the lower level sets.

izations for the uniqueness problem which are, given an optimization problem, not necessarily easy to check.

2.2 Nonconvex Optimization and Necessary Conditions for the Uniqueness Property

The necessary condition for (f, M) to satisfy (P1) requires the notion of a lower semi-continuous correspondence. A correspondence $F : M \rightrightarrows Y$ ³ is *lower semi-continuous* (hereafter *lsc*) at a point $x^* \in M$ if for every $y \in F(x^*)$ and every sequence $\{x_k\} \subset M$ converging to x^* , there exists a positive integer K and a sequence $\{y_k\}$ converging to y , such that

$$y_k \in F(x_k), \quad \forall k \in \{K, K+1, \dots\}.$$

³The *correspondence* F is a point-to-set mapping of points of M into subsets of Y .

When $Y \subset \mathbb{R}^n$, it can be shown that F is lsc at a point $x^* \in M$ if and only if for every open set $A \subset \mathbb{R}^n$ such that

$$A \cap F(x^*) \neq \emptyset,$$

there is an open neighborhood $B(x^*, \delta)$ such that for all $x \in B(x^*, \delta) \cap M$, we have

$$A \cap F(x) \neq \emptyset.$$

The following proposition by Avriel [6] characterizes the necessary and sufficient condition for (f, M) to satisfy (P1). We include the proof for completeness.

Proposition 2.3 Consider a set $M \subset \mathbb{R}^n$ and a function $f : M \mapsto \mathbb{R}$. Let $G_f = \{\alpha \in \mathbb{R} \mid L_\alpha \neq \emptyset\}$. Then, (f, M) satisfies (P1) if and only if $l : G_f \rightrightarrows M$ defined by $l(\alpha) = L_\alpha$, is lsc on G_f .

Proof.

We first prove the if part. Assume that $l(\alpha)$ is lsc, but f does not satisfy (P1). There exists a local minimum $\hat{x} \in M$ of f which is not a global minimum, i.e. there exists $\tilde{x} \in M$ such that

$$f(\tilde{x}) < f(\hat{x}). \quad (2.5)$$

Consider the sequence $\{\alpha_i\}$ defined by

$$\alpha_i = \frac{1}{i}f(\tilde{x}) + \left(1 - \frac{1}{i}\right)f(\hat{x}). \quad (2.6)$$

Clearly,

$$\lim_{i \rightarrow \infty} \alpha_i = f(\hat{x}) = \hat{\alpha}.$$

and $\hat{x} \in l(\hat{\alpha})$. From Equations (2.5) and (2.6), we have

$$f(\tilde{x}) \leq \alpha_i < f(\hat{x}), \quad \forall i \in \{1, 2, \dots\} \quad (2.7)$$

thus $\tilde{x} \in l(\alpha_i)$ for all $i \in \{1, 2, \dots\}$.

Since $l(\alpha)$ is lsc, there exists a natural number K and a sequence $\{x_i\}$ converging to \hat{x} such that $x_i \in l(\alpha_i)$ for $i \in \{K, K+1, \dots\}$. Hence

$$f(x_i) \leq \alpha_i, \quad \forall i \in \{K, K+1, \dots\}$$

and from Eq. (2.7),

$$f(x_i) < f(\hat{x}), \quad \forall i \in \{K, K+1, \dots\}. \quad (2.8)$$

But since $x_i \rightarrow \hat{x}$ and \hat{x} is a local minimum, Eq. (2.8) yields a contradiction. We conclude that f satisfies (P1) whenever $l(\alpha)$ is lsc.

We use the equivalent definition of lower semicontinuity to prove the converse. Assume that (f, M) satisfies (P2) but f is not lsc. Then there is $\hat{\alpha} \in G_f$ such that f is not lsc at $\hat{\alpha}$. This implies there exists an open set $A \subset \mathbb{R}^n$ such that

$$A \cap l(\hat{\alpha}) \neq \emptyset \quad (2.9)$$

and for every $\delta > 0$, there exists $\alpha(\delta) \in B(\alpha, \delta) \cap G_f$ such that

$$A \cap l(\alpha(\delta)) = \emptyset.$$

Therefore, there exists a sequence $\{\alpha_i\} \subset G_f$ converging to $\hat{\alpha}$ such that

$$A \cap l(\alpha_i) = \emptyset, \quad \forall i \in \{1, 2, \dots\}. \quad (2.10)$$

If $\alpha_k \geq \hat{\alpha}$ for some k , then $l(\hat{\alpha}) \subset l(\alpha_k)$ and this yields contradiction in view of (2.9) and (2.10). It follows that $\alpha_i < \hat{\alpha}$ for all i .

Consider $\hat{x} \in A \cap l(\hat{\alpha})$. Then from Eq. (2.10), $\hat{x} \notin l(\alpha_i)$ for all i , and hence

$$f(\hat{x}) > \alpha_i, \quad \forall i \in \{1, 2, \dots\}. \quad (2.11)$$

We deduce that \hat{x} is not a global minimum. Since $\hat{x} \in l(\hat{\alpha})$, we have $f(\hat{x}) \leq \hat{\alpha}$. Then

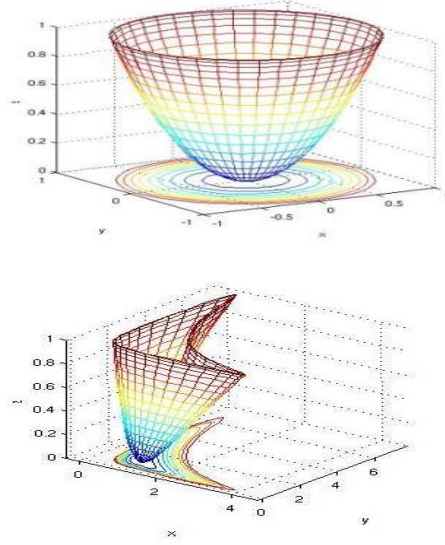


Figure 2.4: In this example, an invertible deformation takes the convex region into a non-convex one while maintaining the characteristics of the critical points.

from Eq. (2.11) and the fact that $\alpha_i \rightarrow \hat{\alpha}$, we have

$$f(\hat{x}) = \hat{\alpha}. \quad (2.12)$$

For every $x \in A \cap M$, (2.10) implies that $x \notin l(\alpha_i)$ for all i , and hence $f(x) > \alpha_i$ for all i . Since $\alpha_i \rightarrow \hat{\alpha}$, it must be the case that $f(x) \geq \hat{\alpha}$. This combined with Eq. (2.12) implies that \hat{x} is a local minimum of f .

$\hat{x} \in M$ is a local minimum of f that is not a global minimum over M . This yields a contradiction since f satisfies (P2). We conclude that f is lsc, completing the proof of the converse statement and the proof of the proposition. **Q.E.D.**

Before we proceed to characterize the necessary conditions for (P2), we note the following example which illustrates the fact that convexity properties may not be well-suited to the analysis of properties of local and global optima of an optimization problem.

Example 2.4 Let M be the unit ball in \mathbb{R}^2 , i.e.,

$$M = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}.$$

Consider the convex function $f : M \mapsto \mathbb{R}$ given by

$$f(x, y) = x^2 + y^2.$$

Consider the function $\phi : M \mapsto \mathbb{R}^2$, a differentiable transformation of the unit ball with a differentiable inverse, given by

$$\phi(x, y) = (e^{x-y}, e^y).$$

Note that $\phi^{-1} : \mathbb{R}_+^2 \mapsto \mathbb{R}^2$ ⁴ is

$$\phi^{-1}(x, y) = (\ln x + \ln y, \ln y).$$

Let $g : \phi(M) \mapsto \mathbb{R}$

$$g(x) = f(\phi^{-1}(x)).$$

Then g also has a unique local minimum, but is neither convex nor quasi-convex as is seen from Figure 2.4.

Example 2.4 demonstrates that it is possible to transform a convex function into a non-convex one by smoothly deforming its domain. Moreover, this process does not change the number and properties of the local extrema (minima or maxima) of the function. This example suggests that the notion of *connectivity by a line within the set* which underpins the notion of convexity is not resistant to differentiable transformations of the domain whereas the properties of extrema are. Hence, in light of example 2.4, the following generalization of quasi-convexity, which uses the notion of *connectivity by a path within the set*, seem more natural.

⁴ \mathbb{R}_+ denotes the set of positive real numbers.

Consider a set $M \subset \mathbb{R}^n$ and a function $f : M \mapsto \mathbb{R}$. f is *connected* on M if, given any $x, y \in M$, there exists a continuous *path* function $p : [0, 1] \mapsto M$ such that $p(0) = x$, $p(1) = y$, and

$$g(p(t)) \leq \max\{g(x), g(y)\}, \quad \forall t \in [0, 1] \quad (2.13)$$

The function f is *strictly connected* on M if, whenever $x \neq y$, the function p can be chosen such that strict inequality holds in (2.13) for all $t \in (0, 1)$.

Given $M \subset \mathbb{R}^n$ and $x, y \in M$, (x, y) is said to be *path-connected* if there exists a continuous function $p : [0, 1] \mapsto S$ such that $p(0) = x$ and $p(1) = y$. Note that path-connectedness is a transitive relation, i.e., if $x, y, z \in M$, (x, y) and (y, z) are path-connected, then (x, z) is also path-connected. M is path-connected if (x, y) is path-connected for any $x, y \in M$. It can be shown that f is connected on M if and only if every non-empty lower level set of f is path-connected⁵.

It can be seen that every quasi-convex [respectively, strictly quasi-convex] function is also connected [respectively, strictly connected], but Example 2.4 shows that the reverse implication is not true. We have the following generalization of Proposition 2.2 due to Ortega-Rheinboldt [41]. We include the proof for completeness.

Proposition 2.4 Consider a set $M \subset \mathbb{R}^n$ and a function $f : M \mapsto \mathbb{R}$. If f is strictly connected on M , then (f, M) satisfies (P1) and (P2).

Proof. First assume that f does not satisfy (P1). Let $x \in M$ be a local minimum that is not a global minimum, i.e. assume that there exists $y \in M$ such that

$$f(y) < f(x).$$

Since f is strictly connected, there is a continuous path function $p : [0, 1] \mapsto M$ such that $p(0) = x$, $p(1) = y$, and

$$f(p(t)) < f(x), \quad \forall t \in (0, 1).$$

⁵See Section 4.2 of Ortega-Rheinboldt [41] for the proof.

Then, for any $\epsilon > 0$, there exists $t \in (0, 1)$ such that $\|p(t) - x\| < \epsilon$ and

$$f(p(t)) < f(x),$$

contradicting the fact that x is a local minimum of f over M . This implies that f satisfies (P1).

Next assume that f does not satisfy (P2). Let x, y be two distinct local minima of f . Without loss of generality, assume that

$$f(y) \leq f(x).$$

Since f is strictly connected, there is a continuous path function $p : [0, 1] \mapsto M$ such that $p(0) = x$, $p(1) = y$, and

$$f(p(t)) < f(x), \quad \forall t \in (0, 1).$$

Then, for any $\epsilon > 0$, there exists $t \in (0, 1)$ such that $\|p(t) - x\| < \epsilon$ and

$$f(p(t)) < f(x),$$

contradicting the fact that x is a local minimum of f over M . We conclude that f satisfies (P2). **Q.E.D.**

We can establish the converse result for Proposition 2.4 only under additional assumptions. Let $M \subset \mathbb{R}^n$ be a path-connected set and $f : M \mapsto \mathbb{R}$ be a function. f is said to be a *well-behaved* function if, for every $\alpha \in \mathbb{R}$ such that the strict lower level set

$$L_\alpha^\circ = \{x \in M \mid f(x) < \alpha\}$$

is path-connected, it follows that $L_\alpha^\circ \cup \{x\}$ is path-connected for every $x \in \text{cl}(L_\alpha^\circ)$.

Avriel et al. [5] demonstrate that most functions of interest are well-behaved, i.e. that restricting to well-behaved functions is a weak assumption. Then, under this

assumption, they prove various converse results to Proposition 2.4 (see Chapter 9 of [5]). We present a partial converse to Proposition 2.4 and refer to [5] for the proof and similar characterizations under different sets of assumptions.

Proposition 2.5 Let $M = \mathbb{R}^n$ and consider a continuous well-behaved function $f : M \mapsto \mathbb{R}$. Assume that the lower level set, L_α , is compact for every $\alpha \in \mathbb{R}$ and that (f, M) satisfies (P2). Then f is strictly connected over M .

Thus, we have partially characterized necessary and sufficient conditions for (f, M) to satisfy (P1) and (P2). However, we note that the results we have presented so far are theoretical characterizations and not conditions that can be checked. In this thesis, our goal is to establish local conditions which, given an optimization problem, can be checked algebraically and which imply (P2) (and hence (P4) under assumptions which assert the existence of a global optimum).

2.3 Characterizing Uniqueness by Local Properties

We proceed by putting more structure on the function f . Let $M \subset \mathbb{R}^n$ be a set and $U \subset \mathbb{R}^n$ be an open set containing M . Let $f : U \mapsto \mathbb{R}$ be a continuously differentiable function. We say that $x \in M$ is a *stationary point* of f over M if $\nabla f(x) = 0$. A classical necessary condition for a point $x \in \text{int}(M)$ to be a local minimum or a local maximum of f over M is $\nabla f(x) = 0$ (see Bertsekas [11], Bazaara-Shetty [13]). We introduce the following property.

(Q1) Every stationary point of f over M is a strict local minimum of f over M .

Note that (Q1) is a local property and the claim only needs to be checked for the stationary points of f . It is tempting to think that (Q1) implies (P2) and (P1). The following proposition shows that this is indeed the case when M is a path-connected subset of \mathbb{R} . However, the subsequent example shows that the claim no longer holds in higher dimensions.

Proposition 2.6 Let $M \subset \mathbb{R}$ be a path-connected set, $U \subset \mathbb{R}$ be an open set containing M , and $f : U \mapsto \mathbb{R}$ be a continuously differentiable function. If (f, M) satisfies (Q1), then (f, M) satisfies (P1) and (P2).

Proof. The path-connected subsets of \mathbb{R} are the open and closed intervals. Thus, $M = [a, b]$ or $M = (a, b)$ where a can represent $-\infty$ and b can represent $+\infty$. We will prove the proposition for the closed interval case, i.e. for $M = [a, b]$. The proof for the case when M is an open interval is identical.

First assume that (f, M) does not satisfy (P1). Then there exists $x, y \in [a, b]$ such that x is a local minimum of f and $f(y) < f(x)$. We note that

$$I_{(x,y)} = [\min(x, y), \max(x, y)] \subset [a, b].$$

Let

$$m = \operatorname{argmax}_{m' \in I_{(x,y)}} f(m') \tag{2.14}$$

Then, $m \neq y$ since $f(y) < f(x)$ and $m \neq x$ since x is a local minimum of f over $[a, b]$ containing $I_{(x,y)}$. Then, $m \in \operatorname{int}(I_{(x,y)})$ and thus $\nabla f(m) = 0$. But then $m \in [a, b]$ is a stationary point of f that is a local maximum (and thus not a strict local minimum), violating (Q1). We conclude that (f, M) satisfies (P1).

Next assume that (f, M) does not satisfy (P2). Then there exists $x, y \in [a, b]$ such that x, y are local minima of f . We have

$$I_{(x,y)} = [\min(x, y), \max(x, y)] \subset [a, b].$$

Let m be defined as in Eq. (2.14). Then, $m \neq x$ (respectively, $m \neq y$) since x (respectively, y) is a local minimum of f over $[a, b]$. We have $m \in \operatorname{int}(I_{(x,y)})$, which further implies that $\nabla f(m) = 0$. But then $m \in [a, b]$ is a stationary point of f that is a local maximum (and thus not a strict local minimum), violating (Q1). We conclude that (f, M) satisfies (P2) as well, completing the proof of Proposition 2.6. **Q.E.D.**

The following example shows that Proposition 2.6 does not generalize to higher

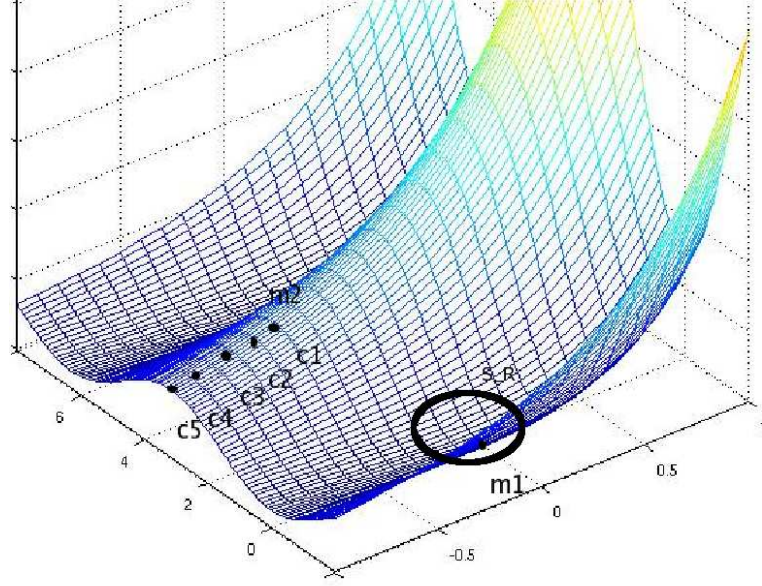


Figure 2.5: In this example, (Q1) is satisfied but (P2) is violated, thus showing that Proposition 2.6 does not generalize to higher dimensions.

dimensions.

Example 2.5 Consider the set $M = \mathbb{R} \times (-2, 8)$ and the function $f : M \mapsto \mathbb{R}$ given by,

$$f(x, y) = (e^x \cos y - 1)^2 - (e^x \sin y)^2.$$

As seen from Figure 2.5, f has exactly two stationary points over M at $m_1 = (0, 0)$ and $m_2 = (0, 2\pi)$, both of which are strict local minima. Then, (f, M) satisfies (Q1) but it does not satisfy (P2) ⁶.

The focus of the remainder of the thesis is to establish conditions along the property (Q1), which are also local in nature, but which allow Proposition 2.6 to hold when M is a subset of a higher dimensional Euclidean space. In Chapter 2, we investigate the mountain pass theory to establish sufficient conditions for the uniqueness of the local minimum when M is all of \mathbb{R}^n or a region defined by finitely many smooth inequality constraints. In Chapter 3, we use the topological index theory to show

⁶In this example, (f, M) satisfies (P1), but it is clear that by slightly perturbing the values of f at the stationary points, one can get an example in which (f, M) satisfies (Q1) but neither (P2) nor (P1).

uniqueness results when M is defined by finitely many smooth inequality constraints. For the constrained region case, the results we get by the two approaches are comparable but not identical. The mountain pass approach asserts the uniqueness of the local minimum by requiring weaker structural assumptions on the function but stronger boundary assumptions than the index theory approach.

Chapter 3

Mountain Pass Theory and the Uniqueness Property

In this chapter, we investigate the mountain pass theory and its implications for establishing sufficient local conditions for the uniqueness of the local minimum. We present the Mountain Pass Theorem of Ambrosetti-Rabinowitz [4] and show that for the case when $M = \mathbb{R}^n$, Proposition 2.6 holds with the additional assumption that f satisfies a compactness-like condition for functions on non-compact domains. Next we present our constrained version of the Mountain Pass Theorem, which consists of two parts. The first part of our result could be derived from a constrained mountain pass theorem by Tintarev [59]. We present a different proof, which allows us to also show the second part of the result, which appears to be new. We then use our theorem to show that for the case when M is a constrained region, Proposition 2.6 holds with the additional assumption that f satisfies certain boundary conditions.

3.1 Mountain Pass for Functions Defined on \mathbb{R}^n

Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a continuously differentiable function. A sequence $\{x_k\} \in \mathbb{R}^n$ such that

$$f(x_k) \rightarrow c \in \mathbb{R}, \quad \nabla f(x_k) \rightarrow 0 \tag{3.1}$$

is called a *Palais-Smale sequence of f* . The function f satisfies the *Palais-Smale condition* on \mathbb{R}^n if the following property holds

(PS) Any Palais-Smale sequence of f over \mathbb{R}^n has a convergent subsequence.

Since every sequence in a compact subset of \mathbb{R}^n has a convergent subsequence, functions defined on compact sets would automatically satisfy (PS). The (PS) condition, then, is a substitute for compactness when the function is defined on a non-compact set (in particular \mathbb{R}^n), which is often used in variational analysis to establish the existence of stationary points (see Jabri [30] for variants and a more detailed analysis). We include the following useful lemma by Jabri [30] for checking the (PS) condition.

Lemma 3.1 Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a continuously differentiable function. Assume that the function

$$g = |f| + \|\nabla f\| : \mathbb{R}^n \mapsto \mathbb{R}$$

is *coercive*, that is, $g(u_k) \rightarrow \infty$ for any sequence $\{u_k\}$ such that $\|u_k\| \rightarrow \infty$. Then f satisfies (PS).

Proof. Let $\{u_k\}$ be a PS sequence for f . Then, the definition in (3.1) and the fact that g is coercive imply that u_k is a bounded sequence. Then, u_k has a convergent subsequence and f satisfies (PS). **Q.E.D.**

The following is the celebrated *Mountain Pass Theorem* by Ambrosetti-Rabinowitz [4] applied to functions defined on finite dimensional spaces ¹.

Theorem 3.1 Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a continuously differentiable function which satisfies the following *mountain pass geometry* property (see Figure 3.1).

(MPG) There exists $r > 0$, $\gamma \in \mathbb{R}$, and $x_1, x_2 \in \mathbb{R}^n$ such that

$$f(u) \geq \gamma, \text{ for every } u \in S(x_1, r) = \{u \in \mathbb{R}^n \mid \|u - x_1\| = r\}$$

¹The original result by Ambrosetti-Rabinowitz also holds for functions on infinite dimensional *Banach spaces*.

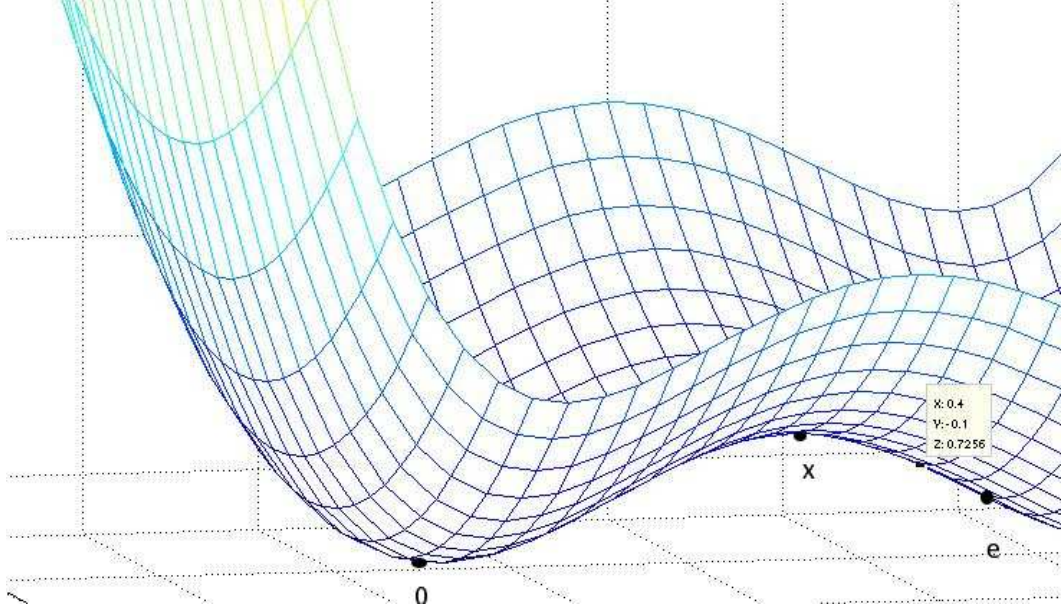


Figure 3.1: The function f satisfies the mountain pass geometry (MPG) property. The situation resembles the problem of a traveller (at point $x_1 = 0$) surrounded by mountains trying to get to a point on the other side of the valley (point $x_2 = e$), motivating the terminology. If the objective of the traveller is to climb the least amount of height, then she has to go through a mountain pass point (point $z = x$), which also turns out to be a stationary point of the function defining the landscape.

and

$$f(x_1) < \gamma, \quad f(x_2) < \gamma, \quad \text{and} \quad \|x_1 - x_2\| > r.$$

(i) Then f has a Palais-Smale sequence $\{x_k\}$ such that

$$f(x_k) \rightarrow c = \inf_{p \in \mathcal{P}} \max_{t \in [0,1]} f(p(t)) \geq \gamma \quad (3.2)$$

where \mathcal{P} denotes the set of continuous functions $p : [0, 1] \mapsto \mathbb{R}^n$ such that $p(0) = x_1$ and $p(1) = x_2$.

(ii) If, in addition, f satisfies (PS), then, f has a stationary point $z \in \mathbb{R}^n$ such that

$$f(z) = c.$$

We call z a *mountain pass stationary point* of f .

We include the proof of Theorem 3.1, which is very insightful. The proof considers the path with the lowest maximum function value ² among all paths connecting x_1 and x_2 , and shows that there is a mountain pass stationary point among the points where this path attains its maximum. If there was not, then one could deform the path at points around maxima along the negative gradient direction to get a path which connected x_1 and x_2 and which had a strictly less maximum function value than the lowest maximum function value path, yielding a contradiction ³.

We first present some preliminary definitions and results. Let $A \subset \mathbb{R}^a$, $B \subset \mathbb{R}^b$ be sets and $f : A \mapsto B$ be a function. f is called *locally Lipschitz continuous* if, for each $x \in A$, there exists an open set $U \subset \mathbb{R}^a$ containing x and $c_x \in \mathbb{R}$ such that

$$\|f(x) - f(y)\| < c_x \|x - y\|$$

for all $y \in U \cap A$. f is called *Lipschitz continuous* if there exists $c \in \mathbb{R}$ such that

$$\|f(x) - f(y)\| < c \|x - y\|$$

for all $x, y \in A$.

For the proof of Theorem 3.1, we need the notion of a pseudo-gradient vector field which acts as a proxy for the gradient that satisfies stronger differentiability assumptions. Given $M \subset \mathbb{R}^n$ and a continuous vector valued function $F : M \mapsto \mathbb{R}^n - \{0\}$, a *pseudo-gradient vector field* for F is a locally Lipschitz continuous function

²We suppose, for the sake of argument, that the path with the lowest maximum value exists. In general, this path might not exist since the set of functions connecting x_1 and x_2 over \mathbb{R}^n is not compact. In that case, the proof considers a sequence of paths converging to the lowest possible maximum value and finds a (PS) sequence. Then the (PS) condition implies the existence of the mountain pass stationary point.

³The deformation along the gradient idea is originally due to Courant [18]. In [18], Courant uses a linear deformation to prove a weaker version of the Mountain Pass Theorem. Ambrosetti-Rabinowitz [4] use, instead, a deformation along the negative gradient flow to prove this version. Since then, various deformation ideas were used in proving (usually infinite dimensional) mountain pass theorems under different sets of assumptions. We refer to Jabri [30] for a good survey of mountain pass results.

$G : M \mapsto \mathbb{R}^n$ such that for every $x \in M$

$$\begin{aligned} \|G(x)\| &\leq 2\|F(x)\| \\ F(x)^T G(x) &\geq \|F(x)\|^2. \end{aligned} \tag{3.3}$$

The following result makes the notion of a pseudo-gradient vector field useful (see Willem [62] for proof).

Lemma 3.2 Consider a set $M \subset \mathbb{R}^n$ and a continuous function $F : M \mapsto \mathbb{R}^n - \{0\}$. Then F has a pseudo-gradient vector field.

We next present a deformation lemma.

Lemma 3.3 Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a continuously differentiable function, $c \in \mathbb{R}$, $\epsilon > 0$. Assume the following

$$\|\nabla f(x)\| \geq \delta > 0, \quad \text{for all } x \in f^{-1}([c - 2\epsilon, c + 2\epsilon]). \tag{3.4}$$

Then there exists a continuous function $\sigma : \mathbb{R}^n \mapsto \mathbb{R}^n$ such that

- (i) $\sigma(x) = x$, for all $x \notin f^{-1}([c - 2\epsilon, c + 2\epsilon])$.
- (ii) $\sigma(L_{c+\epsilon}) \subset L_{c-\epsilon}$ where L_α denotes the lower level set of f at α .

Proof.

Let

$$\begin{aligned} Q_1 &= f^{-1}([c - \epsilon, c + \epsilon]), \\ Q_2 &= f^{-1}([c - 2\epsilon, c + 2\epsilon]). \end{aligned}$$

Given $x \in \mathbb{R}^n$ and a nonempty $A \subset \mathbb{R}^n$, let

$$\text{dist}(x, A) = \inf\{y \in A \mid \|x - y\|\}.$$

Define the function $w : \mathbb{R}^n \mapsto \mathbb{R}$ such that

$$w(x) = \frac{\text{dist}(x, \mathbb{R}^n - Q_2)}{\text{dist}(x, \mathbb{R}^n - Q_2) + \text{dist}(x, Q_1)},$$

then w is a locally Lipschitz continuous function such that $w(x) = 1$ when $x \in Q_1$ and $w(x) = 0$ when $x \notin Q_2$. By Lemma 3.2, there exists a pseudo-gradient vector field G for ∇f on

$$M = \{x \in \mathbb{R}^n \mid \nabla f(x) \neq 0\}.$$

Note that by assumption (3.4), $Q_2 \subset M$. We can then define the locally Lipschitz continuous function $v : \mathbb{R}^n \mapsto \mathbb{R}^n$ with

$$v(x) = \begin{cases} -w(x) \frac{G(x)}{\|G(x)\|^2} & \text{if } x \in Q_2 \\ 0 & \text{if } x \notin Q_2. \end{cases}$$

For $x \in Q_2$, by the definition in (3.3),

$$G(x)^T \nabla f(x) \geq \|\nabla f(x)\|^2$$

which, since $\nabla f(x) \neq 0$, implies

$$\|G(x)\| \geq \|\nabla f(x)\|.$$

Then, using assumption (3.4), we have for $x \in Q_2$

$$\|G(x)\| \geq \|\nabla f(x)\| \geq \delta$$

and hence

$$\|v(x)\| \leq \delta^{-1}, \quad \forall x \in \mathbb{R}^n.$$

Then since the vector valued function v is bounded and locally Lipschitz continuous,

the Cauchy problem

$$\begin{aligned}\frac{\partial}{\partial t}\rho(t, x) &= v(\rho(t, x)), \\ \rho(0, x) &= x,\end{aligned}$$

has a unique solution $\rho(., x) : \mathbb{R} \mapsto \mathbb{R}^n$. Moreover, ρ is continuous on $\mathbb{R} \times \mathbb{R}^n$. Let $\sigma : \mathbb{R}^n \mapsto \mathbb{R}$ be

$$\sigma(x) = \rho(8\epsilon, x).$$

Then σ is a continuous function. For $x \notin Q_2$, $v(x) = 0$ and thus $\rho(t, x) = x$ for all $t > 0$. In particular, $\sigma(x) = \rho(8\epsilon, x) = x$ and σ satisfies (i). For $x \in Q_2$, we have

$$\begin{aligned}\frac{\partial}{\partial t}f(\rho(t, x)) &= \nabla f(\rho(t, x))^T \frac{\partial}{\partial t}\rho(t, x) \\ &= \nabla f(\rho(t, x))^T v(\rho(t, x)) \\ &= -w(\rho(t, x)) \frac{\nabla f(\rho(t, x))^T G(\rho(t, x))}{\|G(\rho(t, x))\|^2}\end{aligned}\tag{3.5}$$

$$\leq -\frac{w(\rho(t, x))}{4}\tag{3.6}$$

where we used the definition in (3.3) to get the inequality. Thus, $f(\rho(., x)) : \mathbb{R} \mapsto \mathbb{R}$ is a non-increasing function. Let $x \in L_{c+\epsilon}$. If there exists $t \in [0, 8\epsilon]$ such that $f(\rho(t, x)) < c - \epsilon$, then $f(\rho(8\epsilon, x)) < c - \epsilon$ and σ satisfies (ii) for x . Otherwise,

$$\rho(t, x) \in Q_1, \quad \forall t \in [0, 8\epsilon].$$

Then using Eq. (3.6) and the fact that $w(x) = 1$ for $x \in Q_1$, we have

$$\begin{aligned}
f(\rho(8\epsilon, x)) &= f(x) + \int_0^{8\epsilon} \frac{\partial}{\partial t} f(\rho(t, x)) dt \\
&\leq f(x) - \int_0^{8\epsilon} \frac{w(\rho(t, x))}{4} dt \\
&= f(x) - \frac{8\epsilon}{4} \\
&\leq c + \epsilon - 2\epsilon = c - \epsilon,
\end{aligned}$$

Thus σ satisfies (ii) as well. This completes the proof of the lemma. **Q.E.D.**

Proof of Theorem 3.1. Consider a path $p \in \mathcal{P}$. $p(0) = x_1$, $p(1) = x_2$, and therefore from continuity of p ,

$$p([0, 1]) \cap S(x_1, r) \neq \emptyset.$$

Hence, by definition of c in (3.2) and by the (MPG) assumption, we have

$$c \geq \gamma > \max(f(x_1), f(x_2)).$$

Let $\epsilon' = c - \max(f(x_1), f(x_2)) > 0$. We claim that for all $\epsilon \in \mathbb{R}$ such that $\epsilon' > 2\epsilon > 0$, there exists $u_\epsilon \in \mathbb{R}^n$ such that

$$\begin{aligned}
c - 2\epsilon &\leq f(u_\epsilon) \leq c + 2\epsilon, \quad \text{and} \\
\|\nabla f(u_\epsilon)\| &< 2\epsilon.
\end{aligned} \tag{3.7}$$

Suppose for some positive $\epsilon < \epsilon'/2$, there is no $u_\epsilon \in \mathbb{R}^n$ satisfying the conditions in (3.7). Then f satisfies the assumption of Lemma 3.3 with $\delta = 2\epsilon$ and there exists a continuous deformation function $\sigma : \mathbb{R}^n \mapsto \mathbb{R}^n$ that satisfies conditions (i) and (ii) of Lemma 3.3. By definition of c in (3.2), there exists $p \in \mathcal{P}$ such that

$$\max_{t \in [0, 1]} f(p(t)) \leq c + \epsilon.$$

Define $q : [0, 1] \mapsto \mathbb{R}^n$ with

$$q(t) = \sigma(p(t)).$$

We note that since $\epsilon < \epsilon'/2$,

$$f(x_1) \leq c - 2\epsilon,$$

thus $f(x_1) \notin f^{-1}([c - 2\epsilon, c + 2\epsilon])$ and by condition (i) of Lemma 3.3

$$q(0) = \sigma(f(0)) = \sigma(x_1) = x_1$$

and similarly $q(1) = x_2$. Since q is also continuous, we have $q \in \mathcal{P}$. Since $p(t) \in L_{c+\epsilon}$ for all $t \in [0, 1]$, by condition (ii) of Lemma 3.3, we have

$$\max_{t \in [0, 1]} f(q(t)) \leq c - \epsilon < c$$

which contradicts the definition of c . Thus, we conclude that for each positive $\epsilon < \epsilon'/2$ there exists u_ϵ satisfying (3.7). For each $k \in \mathbb{Z}^+$, let

$$x_k = u_{\epsilon'/2^{k+2}}.$$

Then $\nabla f(x_k) \rightarrow 0$ and $f(x_k) \rightarrow c$, thus $\{x_k\}$ is a Palais-Smale sequence satisfying (3.2) as desired.

Now assume that f also satisfies (PS). Then x_k has a convergent subsequence that converges to, say $z \in \mathbb{R}$. By the continuity of ∇f and f , we have

$$\nabla f(z) = 0, \quad \text{and} \quad f(z) = c$$

thus z is a mountain pass stationary point, completing the proof of Theorem 3.1.

Q.E.D.

Given a continuously differentiable function $f : \mathbb{R}^n \mapsto \mathbb{R}$, let

$$K_c(f) = \{z \in \mathbb{R}^n \mid f(z) = c, \nabla f(z) = 0\}.$$

Let $c \in \mathbb{R}$ be the constant defined in (3.2). Then $K_c(f)$ denotes the set of mountain pass stationary points of f , which is non-empty by Theorem 3.1. It is also of interest to understand the structure of $K_c(f)$. The construction in the proof of Theorem 3.1 suggests that, except for degenerate cases, $z \in K_c(f)$ would be a *saddle point*, i.e. a stationary point that is neither a minimum nor a maximum. The following result by Pucci-Serrin [42] makes this idea more precise (see [42] for the proof).

Theorem 3.2 Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a function that satisfies the assumptions of Theorem 3.1. Then there exists either a saddle point or a strict local maximum of f in $K_c(f)$ where $c \in \mathbb{R}$ denotes the constant defined in (3.2).

We next investigate the implications of these results on the uniqueness problem. Theorems 3.1 and 3.2 allow us to generalize Proposition 2.6.

Proposition 3.1 Let $M = \mathbb{R}^n$, and $f : M \mapsto \mathbb{R}$ be a continuously differentiable function. If (f, M) satisfies (Q1) and f satisfies (PS), then (f, M) satisfies (P2).

Proof. Assume the contrary, that (f, M) does not satisfy (P2). Then there exists $x_1, x_2 \in \mathbb{R}^n$ such that x_1, x_2 are distinct local minima of f over \mathbb{R}^n . From the first order optimality conditions, x_1, x_2 are stationary points of f over \mathbb{R}^n . Then since (f, M) satisfies (Q1), we have that x_1, x_2 are strict local minima.

Without loss of generality assume $f(x_2) \leq f(x_1)$. Since x_1 is a strict local minimum, there exists a sufficiently small $\epsilon > 0$ such that $\epsilon < \|x_1 - x_2\|$ and

$$f(u) > f(x_1) \text{ for all } u \in S(x_1, \epsilon). \quad (3.8)$$

Let

$$\gamma = \min_{u \in S(x_1, \epsilon)} f(u).$$

Then from (3.8), we have

$$f(x_2) \leq f(x_1) < \gamma$$

which implies that f satisfies the property (MPG) of Theorem 3.1. Then since f also satisfies (PS), f has a mountain pass stationary point $z \in \mathbb{R}^n$. From Theorem 3.2,

z can be chosen such that it is either a saddle point or a strict local maximum of f . Then, z is a stationary point that is not a strict local minimum, contradicting the fact that f satisfies (Q1). Therefore, we conclude that (f, M) satisfies (P2), completing the proof of Proposition 3.1. **Q.E.D.**

We illustrate the result with the following examples.

Example 3.1 We consider the function in Example 2.5. We see that Proposition 3.1 does not apply since f fails to satisfy (PS). Theorem 3.1 requires that f has a Palais-Smale sequence, which is the sequence $\{c_k\}$ displayed in Figure 2.5. However, $\{c_k\}$ does not have a convergent subsequence and f does not have a mountain pass stationary point. This example shows that the assumption (PS) in Proposition 3.1 cannot be dispensed away. In the absence of the (PS) condition, the valley of the mountain pass geometry can become flat at infinity and the mountain pass stationary point might fail to exist.

Example 3.2 We consider a simple case when convexity properties (Propositions 2.1 and 2.2) cannot be used to assert uniqueness but Proposition 3.1 can. Let $M = \mathbb{R}^2$ and consider the function in Example 2.3 with a slight difference, i.e. let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ such that

$$f(x, y) = -e^{-x^2} + x^2/100 + y^2. \quad (3.9)$$

f is not quasi-convex as clearly seen from Figure 3.2. For $(x, y) \in \mathbb{R}^2$, we have

$$\nabla f(x, y) = [2xe^{-x^2} + x/50, 2y]^T$$

which implies that f has a unique stationary point at $(x^*, y^*) = (0, 0)$. By (3.9), (x^*, y^*) is seen to be a strict global minimum⁴. Thus (f, M) satisfies (Q1). We also claim that f satisfies (PS). Note that we have

$$\lim_{|x| \rightarrow \infty} -e^{-x^2} + x^2/100 = \infty$$

⁴In general, we would use the second order sufficient optimality conditions to show that every stationary point is a strict local minimum.

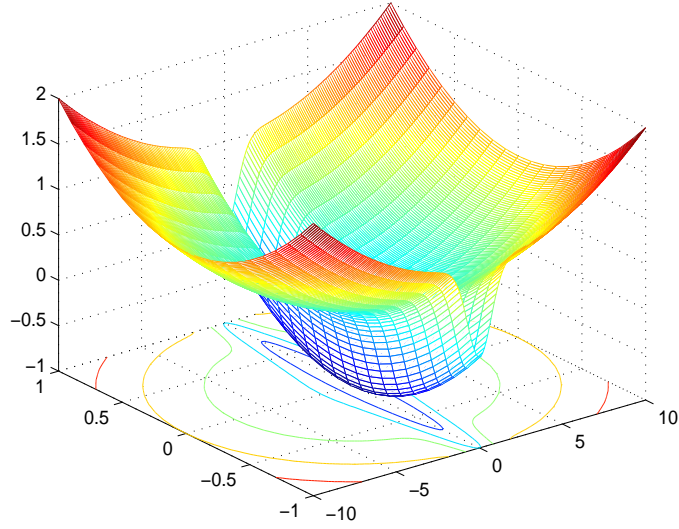


Figure 3.2: The function $-e^{-x^2} + x^2/100 + y^2$ is not quasi-convex, as can be seen from the lower level sets. However, it satisfies (Q1) and (PS), thus Proposition 3.1 applies and it has a unique local (global) minimum.

and

$$\lim_{|y| \rightarrow \infty} y^2 = \infty.$$

Then, the function $|f|$ is coercive, which, by Lemma 3.1, implies that f satisfies (PS). Then, Proposition 3.1 applies to (f, M) and we have that f satisfies (P2). Since a global minimum also exists (f satisfies (P3)), we conclude that f satisfies (P4).

Example 3.3 In, Example 3.1, we slightly modified the function in Example 2.3 to make it satisfy the (PS) condition. Consider the function in Example 2.3 unmodified, i.e. let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be given by

$$f(x, y) = -e^{-x^2} + y^2.$$

Consider the sequence $\{u_n\} \subset \mathbb{R}^n$ defined as

$$u_n = (n, 0), \quad \forall n \in \mathbb{Z}^+.$$

Then, $f(u_n) \rightarrow 0$ and $\nabla f(u_n) \rightarrow 0$. Thus, u_n is a PS sequence which does not have a

convergent subsequence. Then f does not satisfy (PS) and Proposition 3.1 does not apply for f . However, f is clearly seen to satisfy (P2) and (P4) from Figure 2.3. This example shows that the converse to Proposition 3.1 does not necessarily hold and the (PS) assumption may possibly be relaxed further, if not completely dispensed away.

3.2 Mountain Pass for Functions Defined on Constrained Regions

We now consider the case when $M \neq \mathbb{R}^n$. In this case, additional boundary conditions are required for mountain pass arguments to work. We consider regions defined by inequality constraints to be able to specify the boundary conditions. Let M be a non-empty region, defined by finitely many inequality constraints, i.e.,

$$M = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, \ i \in I = \{1, 2, \dots, |I|\}\}, \quad (3.10)$$

where the $g_i : \mathbb{R}^n \mapsto \mathbb{R}$, $i \in I$, are twice continuously differentiable. For some $x \in M$, let $I(x) = \{i \in I \mid g_i(x) = 0\}$ denote the set of active constraints. We assume that every $x \in M$ satisfies the *linear independence constraint qualification* (see [12], Section 5.4), i.e., for every $x \in M$, the vectors $\{\nabla g_i(x) \mid i \in I(x)\}$ are linearly independent. We note that the LICQ condition implies

$$\text{bd}(M) = \{x \in M \mid I(x) \neq \emptyset\}$$

and

$$\text{int}(M) = \{x \in M \mid I(x) = \emptyset\}.$$

We introduce the following boundary condition

$$\text{(B1)} \quad \nabla g_i(x)^T \nabla f(x) > 0, \quad \forall x \in \text{bd}(M), \ i \in I(x).$$

Intuitively, (B1) says that given a point x on the boundary of M , the gradient of f at x points outward M . This condition is required to make sure that the path

deformation in the negative gradient direction used in the mountain pass argument does not leave the region M .

Our constrained mountain pass theorem requires the topological notion of a connected set. Given sets $A, C \subset \mathbb{R}^n$ such that $A \subset C$, we say that A is *open in C* if there exists an open set $U \subset \mathbb{R}^n$ such that $A = C \cap U$. A set $C \subset \mathbb{R}^n$ is said to be a *connected set* if there does not exist non-empty sets $A, B \subset C$ such that A and B are open in C and $C = A \cup B$. Equivalently, C is connected if for any non-empty sets $A, B \subset C$, such that $C = A \cup B$, either

$$A \cap \text{cl}(B) \neq \emptyset, \text{ or } \text{cl}(A) \cap B \neq \emptyset. \quad (3.11)$$

We note that every path-connected set is connected, but the reverse implication is not true.

The following is our version of a constrained mountain pass theorem, which is tailored for optimization problems and which consists of two parts. The first part of the result shows the existence of a mountain pass stationary point and could be derived from a constrained mountain pass theorem by Tintarev [59], which proves the result for functions on infinite dimensional spaces defined by Lipschitz inequality constraints. The second part of the result partially characterizes the set of the mountain pass stationary points, similar to the refinement by Pucci-Serrin [42] [cf. Theorem 3.2] to Theorem 3.1. To our knowledge, the second part is new, and cannot be obtained from similar constrained mountain pass results by Tintarev [57, 58, 59] and Schechter [52, 53].

Theorem 3.3 Let M be a compact and connected region given by (3.10), $U \subset \mathbb{R}^n$ be an open set containing M and $f : U \mapsto \mathbb{R}$ be a continuously differentiable function. Assume that (f, M) satisfies (B1) and the following constrained mountain pass geometry property.

(CMPG) There exists $r > 0$, $\gamma \in \mathbb{R}$, and $x_1, x_2 \in M$ such that

$$f(u) \geq \gamma, \text{ for every } u \in S(x_1, r) \cap M$$

and

$$f(x_1) < \gamma, \quad f(x_2) < \gamma, \quad \text{and} \quad \|x_1 - x_2\| > r.$$

(i) Then, f has a stationary point $z \in M$ such that

$$f(z) = d = \inf_{\Sigma \in \Gamma} \max_{x \in \Sigma} f(x) \geq \gamma \quad (3.12)$$

where

$$\Gamma = \{\Sigma \subset M \mid \Sigma \text{ is compact and connected, } x_1, x_2 \in \Sigma\}.$$

(ii) z can be chosen as not to be a local minimum of f over M .

In proving their results, Tintarev and Schechter use the deformation along the pseudo-gradient vector flow idea that we presented in the proof of Theorem 3.1. In contrast, we use in our proof the linear deformation idea of Courant [18]. The linear deformation approach uses more elementary tools and is more insightful. It also proves more useful in the finite dimensional optimization framework we are interested in. We need the following lemma for the line deformation not to leave the region M .

Lemma 3.4 Let M be a non-empty, compact region given by (3.10). Let $U \subset \mathbb{R}^n$ be an open set containing M and $f : U \mapsto \mathbb{R}$ be a continuously differentiable function. Assume that (f, M) satisfies (B1). Then there exists $T > 0$ such that $x - t\nabla f(x) \in M$ for all $x \in M$ and $t \in [0, T]$.

Proof. Given $i \in I$, define $h_i : M \times \mathbb{R} \mapsto \mathbb{R}$ as

$$h_i(x, t) = \max \left(-g_i(x - t\nabla f(x)), \nabla f(x)^T \nabla g_i(x - t\nabla f(x)) \right).$$

Then

$$h_i(x, 0) = \min(-g_i(x), \nabla f(x)^T \nabla g_i(x)) > 0$$

holds for all $x \in M$, since either $g_i(x) < 0$, or if $g_i(x) = 0$ then from (B1)

$$\nabla f(x)^T \nabla g_i(x) > 0.$$

Since h_i is continuous and M is compact, there exists scalars $\alpha_i > 0$, $\tau_i > 0$ such that

$$h_i(x, t) > \alpha_i, \quad \forall x \in M, t \in [-\tau_i, \tau_i]. \quad (3.13)$$

Let

$$m_{\nabla f} = \max_{x \in M} \|\nabla f(x)\| > 0$$

and

$$M_i = \{y \in R^n \mid \exists x \in M \text{ such that } \|x - y\| \leq m_{\nabla f} \tau_i\}.$$

Clearly, $M \subset M_i$ and M_i is compact. Define the error function $e_i : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ as

$$e_i(x, y) = \begin{cases} \frac{g_i(y) - g_i(x) - \nabla g_i(x)^T (y - x)}{\|y - x\|} & \text{if } y \neq x \\ 0 & \text{if } y = x \end{cases}$$

e_i is continuous on $R^n \times R^n$ since g_i is continuously differentiable. Since $M_i \times M_i$ is compact, e_i is uniformly continuous on $M_i \times M_i$, therefore there exists $\tau'_i > 0$ such that

$$|e_i(x, y)| < \frac{\alpha_i}{m_{\nabla f}}, \quad \forall x, y \text{ with } \|x - y\| \leq \tau'_i m_{\nabla f}. \quad (3.14)$$

Let $T_i = \min(\tau_i, \tau'_i)$. We claim $g_i(x - t\nabla f(x)) < 0$ for all $x \in M$ and $t \in (0, T_i]$. Assume the contrary, that there exists $t \in (0, T_i]$ such that $g_i(x - t\nabla f(x)) \geq 0$. By Eq. (3.13), we have

$$\nabla f(x)^T \nabla(g_i(x - t\nabla f(x))) > \alpha_i > 0. \quad (3.15)$$

From the first order Taylor's approximation for g_i at $x - t\nabla f(x)$ with direction $t\nabla f(x)$ we have

$$\begin{aligned} g_i(x) &= g_i(x - t\nabla f(x)) + t\nabla f(x)^T \nabla g_i(x - t\nabla f(x)) \\ &\quad + t\|\nabla f(x)\|e_i(x - t\nabla f(x), x). \end{aligned}$$

Dividing each side of the equation by $t\|\nabla f(x)\|$ and rearranging, we get

$$e_i(x - t\nabla f(x), x) = \frac{g_i(x) - g_i(x + t\nabla f(x))}{t\|\nabla f(x)\|} - \frac{\nabla f(x)^T \nabla g_i(x - t\nabla f(x))}{\|\nabla f(x)\|}$$

Since $g_i(x) \leq 0$ and by assumption $g_i(x - t\nabla f(x)) \geq 0$, the first term on the right is non-positive. From Eq. (3.15), the second term is bounded above by $\frac{-\alpha_i}{\nabla f(x)}$, thus we have

$$e_i(x - t\nabla f(x), x) \leq \frac{-\alpha_i}{\nabla f(x)} \leq -\frac{\alpha_i}{m_{\nabla f}}. \quad (3.16)$$

However, $x \in M_i$ and $x + t\nabla f(x) \in M_i$ and also

$$\|x - t\nabla f(x) - x\| = t\|\nabla f(x)\| \leq \tau'_i m_{\nabla f},$$

thus Equations (3.14) and (3.16) yield a contradiction. Therefore it must be true that $g_i(x - t\nabla f(x)) < 0$ for all $x \in M$ and $t \in (0, T_i]$.

Let $T = \min_{i \in I} T_i > 0$. Then, for all $x \in M$ and $t \in (0, T]$, $g_i(x + t\nabla f(x)) < 0$ for all $i \in I$, which implies $x + t\nabla f(x) \in M$. This concludes the proof of Lemma 3.4. **Q.E.D.**

Proof of Theorem 3.3 The proof proceeds in a number of steps:

Step 1: Note that the set M is compact, connected, and it contains x_1 and x_2 , thus $M \in \Gamma$ and Γ is non-empty. Let

$$d = \inf_{\Sigma \in \Gamma} \max_{x \in \Sigma} f(x). \quad (3.17)$$

In this step, we show that d is attained as the maximum of f over some $\Sigma \in \Gamma$.

For $\Sigma \in \Gamma$, let d_Σ be given by

$$d_\Sigma = \max_{x \in \Sigma} f(x).$$

Consider a sequence of sets $\{\Sigma_k\} \subset \Gamma$ such that

$$\lim_{k \rightarrow \infty} d_{\Sigma_k} = d.$$

Consider the outer limit of the sequence $\{\Sigma_k\}$,

$$\Sigma = \limsup_{k \rightarrow \infty} \Sigma_k = \bigcap_{m \in \mathbb{Z}^+} \text{cl}\left(\bigcup_{i \geq m} \Sigma_i\right).$$

We claim that $\Sigma \in \Gamma$. Clearly, $x_1 \in \Sigma$ since $x_1 \in \Sigma_i$ for all i . Similarly $x_2 \in \Sigma$. Since each Σ_i is connected and $\bigcap_i \Sigma_i$ is non-empty, $\text{cl}(\bigcup_{i \geq m} \Sigma_i)$ is connected for each m . Then Σ is also connected⁵. Also, since $\text{cl}(\bigcup_{i \geq m} \Sigma_i)$ is a compact subset of M for each m , Σ is a compact subset of M , proving that $\Sigma \in \Gamma$.

We claim $d_\Sigma = d$. Let $x_\Sigma \in \Sigma$ such that $f(x_\Sigma) = d_\Sigma$. Since Σ is the outer limit, every neighborhood of x_Σ has non-empty intersection with infinitely many Σ_k 's (see Ozdaglar [39]). Therefore, there exists a sequence of points $\{x_{\Sigma_k}\}$ such that $x_{\Sigma_k} \in \Sigma_k$ for each k , and

$$\lim_{k \rightarrow \infty} x_{\Sigma_k} = x_\Sigma.$$

Then, we have:

$$\begin{aligned} f(x_\Sigma) &= \lim_{k \rightarrow \infty} f(x_{\Sigma_k}) \\ &\leq \lim_{k \rightarrow \infty} d_{\Sigma_k} \\ &= d. \end{aligned}$$

On the other hand, $d_\Sigma \geq d$ since $\Sigma \in \Gamma$, proving that $d_\Sigma = d$ and that the infimum is attained for Σ .

Step 2: Let $P = \text{argmax}_{x \in \Sigma} f(x)$, where Σ is the set constructed in *Step 1*. In this step, we prove the existence of a stationary point in P . Since Σ is connected, contains

⁵It can be shown that the intersection $C = \bigcap_{i=1}^{\infty} C_i$, $C_{i+1} \subset C_i$ is connected if all C_i are connected compact sets.

x_1 and x_2 , and $\|x_1 - x_2\| > r$, we have

$$\Sigma \cap S(x_1, r) \neq \emptyset.$$

Therefore,

$$d = \max_{x \in \Sigma} f(x) \geq \max_{x \in \Sigma \cap S(x_1, r)} f(x) \geq \gamma.$$

Thus

$$d > \max(f(x_1), f(x_2))$$

and $x_1, x_2 \notin P$. Let

$$P^\epsilon = \{y \in M \mid \exists x \in M \text{ such that } \|x - y\| \leq \epsilon\}.$$

Since P is compact and $x_1, x_2 \notin P$, there exists $\epsilon' > 0$ such that $x_1, x_2 \notin P^{\epsilon'}$.

We claim that there exists a point $x \in P$ for which $\nabla f(x) = 0$. Assume the contrary, that for all $x \in P$, $\|\nabla f(x)\| > 0$. Since f is continuous and P is compact, there exists a positive $\epsilon < \epsilon'$ and a positive α such that

$$\|\nabla(f(x))\| > \alpha, \quad \forall x \in P^\epsilon. \quad (3.18)$$

Consider a continuous weight function $w : M \mapsto [0, 1]$ such that

$$w(x) = \begin{cases} 1 & \text{for } x \in P, \\ 0 & \text{for } x \in M - P^\epsilon. \end{cases}$$

Define the deformation $\nu : M \times \mathbb{R} \mapsto \mathbb{R}^n$ by ⁶

$$\nu(x, t) = x - tw(x)\nabla f(x)$$

Let $T > 0$ be sufficiently small that the claim of Lemma 3.4 holds. Then $\nu(x, t) \in$

⁶Compare this with the deformation along the (opposite of) pseudo-gradient flow utilized in the proof of Theorem 3.1.

M for $x \in M$ and $t \in [0, T]$. Define $k : P^\epsilon \times [0, T] \mapsto \mathbb{R}$ as

$$k(x, t) = \frac{\nabla f(\nu(x, t))^T \nabla f(x)}{\|\nabla f(x)\|^2}$$

$k(x, t)$ is well defined since $\|\nabla f(x)\| > \alpha > 0$ for $x \in P^\epsilon$. Moreover, $k(x, t)$ is uniformly continuous since $P^\epsilon \times [0, T]$ is compact. Therefore, since $k(x, 0) = 1$, there exists $T' \in (0, T]$ such that

$$k(x, t) > 1/2, \quad \forall x \in P^\epsilon \text{ and } t \in (0, T']. \quad (3.19)$$

Define the deformed set

$$\Sigma_{T'} = \{\nu(x, T') \mid x \in \Sigma\}.$$

Then, we have $\Sigma_{T'} \subset M$ from Lemma 3.4. Moreover, from continuity of ν , $\Sigma_{T'}$ is compact and connected. $x_1 \in \Sigma_{T'}$ since $\nu(x_1, T') = x_1$, and similarly $x_2 \in \Sigma_{T'}$. Therefore, $\Sigma_{T'}$ is a compact connected set containing x_1 and x_2 and hence $\Sigma_{T'} \in \Gamma$.

We claim

$$f(\nu(x, T')) < d, \quad \text{for all } x \in \Sigma. \quad (3.20)$$

If $w(x) = 0$, $x \notin P$ so

$$f(\nu(x, T')) = f(x) < d.$$

Assume $w(x) > 0$, i.e. that $x \in P^\epsilon$. $f(\nu(x, t))$ is continuously differentiable in t for $t \in (0, T']$ with derivative

$$\begin{aligned} \frac{\partial}{\partial t} f(\nu(x, t)) &= -\nabla f(\nu(x, t))^T w(x) \nabla f(x) \\ &= -w(x) k(x, t) \|\nabla f(x)\|^2. \end{aligned}$$

Then we have

$$\begin{aligned} f(\nu(x, T')) &= f(x) + \int_0^{T'} \frac{\partial}{\partial t} f(\nu(x, t)) dt \\ &= f(x) + \int_0^{T'} -w(x)k(x, t) \|\nabla f(x)\|^2 dx \end{aligned}$$

from which by Equations (3.18) and (3.19), we have

$$f(\nu(x, T')) \leq f(x) - \int_0^{T'} \frac{1}{2} \alpha^2 w(x) dx.$$

If $x \in P^c - P$,

$$f(\nu(x, T')) \leq f(x) < d.$$

Else if $x \in P$, $w(x) = 1$ and

$$f(\nu(x, T')) \leq d - \frac{T'}{2} \alpha^2 < d.$$

Therefore, we conclude that the claim in (3.20) holds.

It follows that

$$\max_{x \in \Sigma_{T'}} f(x) < d,$$

which contradicts the definition in (3.17). Therefore we conclude that there exists $x \in P$ such that $\nabla f(x) = 0$. This concludes the proof of *Step 2* and part (i) of the theorem.

Step 3: We have shown in *Step 2* that the set

$$K_d(f) = \{x \in \Sigma \mid f(x) = d, \nabla f(x) = 0\}.$$

is non-empty. In this step, we prove that there exists $z \in K_d(f)$ such that z is not a local minimum of f . $K_d(f)$ is compact since the set Σ is compact and the functions

f and ∇f are continuous. Then $K_d(f)$ is a closed set and

$$\text{cl}(K_d(f)) = K_d(f). \quad (3.21)$$

Note that $\Sigma - K_d(f)$ and $K_d(f)$ are both non-empty and Σ is connected. Then since

$$K_d(f) \cup (\Sigma - K_d(f)) = \Sigma,$$

and since

$$\text{cl}(K_d(f)) \cap (\Sigma - K_d(f)) = K_d(f) \cap (\Sigma - K_d(f)) = \emptyset,$$

by the definition in (3.11), we have

$$K_d(f) \cap \text{cl}(\Sigma - K_d(f)) \neq \emptyset,$$

i.e. there exists $z \in K_d(f)$ such that $z \in \text{cl}(\Sigma - K_d(f))$. Since (f, M) satisfies (B1), $\nabla f(x) \neq 0$ for $x \in \text{bd}(M)$, which implies that $z \in \text{int}(M)$. We claim that $f(x) < f(z)$ for $x \in M$ arbitrarily close to z . Since $z \in \text{cl}(\Sigma - K_d(f))$, there exists $y \in \text{int}(M) \cap (\Sigma - K_d(f))$ arbitrarily close to z . We have

$$f(y) \leq f(z) = d$$

since d is the maximum of f over Σ . If $f(y) < d$ taking $x = y$ is sufficient. Otherwise, $f(y) = d$ but $\nabla f(y) \neq 0$ since $y \notin K_d(f)$. Since $y \in \text{int}(M)$ and f is differentiable at y , $-\nabla f(y) \neq 0$ is a descent direction of f at y . Therefore, for arbitrarily small $\tau > 0$,

$$y - \tau \nabla f(y) \in \text{int}(M)$$

and

$$f(y - \tau \nabla f(y)) < d.$$

Taking $x = y - \tau \nabla f(y)$, we find x arbitrarily close to y (and hence to z) such that $f(x) < f(z)$. Thus z is not a local minimum of f , completing the proof of *Step 3* and

part (ii) of the theorem. **Q.E.D.**

We next investigate the implications of the constrained mountain pass theorem on the uniqueness problem. Theorem 3.3 allows us to prove the following generalization of Proposition 2.6 for the case when M is a constrained subset of \mathbb{R}^n , the same way Theorem 3.1 allowed us to prove Proposition 3.1.

Proposition 3.2 Let M be a non-empty, compact, and connected region given by (3.10), $U \subset \mathbb{R}^n$ be an open set containing M and $f : U \mapsto \mathbb{R}$ be a continuously differentiable function. If (f, M) satisfies (Q1) and (B1), then (f, M) satisfies (P2).

Proof. Assume the contrary, that (f, M) does not satisfy (P2). Then there exists $x_1, x_2 \in \mathbb{R}^n$ such that x_1, x_2 are distinct local minima of f over \mathbb{R}^n . From optimality conditions, x_1, x_2 are stationary points of f over \mathbb{R}^n . Then since (f, M) satisfies (Q1), we have that x_1, x_2 are strict local minima. Then f satisfies the property (CPGM) of Theorem 3.3, as shown in the proof of Proposition 3.1. Since (f, M) also satisfies (B1), Theorem 3.3 applies and f has a mountain pass stationary point $z \in \mathbb{R}^n$ which is not a strict local minimum of f over M . This contradicts (Q1) and we conclude that (f, M) satisfies (P2). **Q.E.D.**

We illustrate this result with the following example.

Example 3.4 We consider a function defined on a bounded region where convexity properties does not help in showing uniqueness but our results apply. Consider the deformed function of Example 2.4, i.e., let

$$M = \{(x, y) \in \mathbb{R}^2 \mid x, y > 0, (\ln x + \ln y)^2 + (\ln y)^2 \leq 1\}$$

and consider the function $f : M \mapsto \mathbb{R}$ given by

$$f(x, y) = (\ln x + \ln y)^2 + (\ln y)^2. \quad (3.22)$$

It is seen from Figure 2.4 that f is not quasi-convex. It can be seen that M is defined

as in (3.10) with $I = \{1\}$ and $g_1 : \mathbb{R}_+^2 \mapsto \mathbb{R}$ ⁷

$$g_1(x, y) = (\ln x + \ln y)^2 + (\ln y)^2 - 1$$

since

$$\nabla g_1(x, y) = \left[\frac{2 \ln x + 2 \ln y}{x}, \frac{2 \ln x + 4 \ln y}{y} \right]^T \quad (3.23)$$

is non-zero for all x with $g_1(x, y) = 0$. Note that

$$\nabla f(x, y) = \nabla g_1(x, y). \quad (3.24)$$

Consider $(x, y) \in \text{bd}(M)$. Then since $\nabla g_1(x, y) \neq 0$,

$$\nabla g_1(x)^T \nabla f(x) = \|\nabla g_1(x)\|^2 > 0,$$

showing that (f, M) satisfies (B1). From Equations (3.23) and (3.24), f has a unique stationary point at $(x, y) = (1, 1)$ which, by Eq. (3.22), is a strict global (and local) minimum⁸. Then (f, M) satisfies (Q1) and Proposition 3.2 applies. We conclude that f satisfies (P2) (and thus (P4)), i.e., f has a unique local (global) minimum.

Let M be a region given by (3.10). We introduce a boundary condition which is weaker than (B1).

(B2) $-\nabla f(x) \notin N_M(x)$ for all $x \in \text{bd}(M)$

where $N_M(x)$ denotes the *normal cone*⁹ for the region M , i.e.

$$\begin{aligned} N_M(x) &= \text{conv}\{\nabla g_i(x) \mid i \in I(x)\} \\ &= \left\{ v \in \mathbb{R}^n \mid v = \sum_{i \in I(x)} \lambda_i \nabla g_i(x), \lambda_i \in \mathbb{R}, \lambda_i \geq 0, \forall i \in I(x) \right\}. \end{aligned}$$

⁷Note that the characterization here is not exactly that of (3.10), since g_1 is not defined over all of \mathbb{R}^2 . Since $M \subset \mathbb{R}_+^2$ and g_1 is defined on the relevant region, all of our results apply to this case. In essence, it is possible to cut off g_1 (the log function) for the irrelevant region and replace with a differentiable extension that is defined over all of \mathbb{R}^2 .

⁸In general, we would use the second order sufficient optimality conditions to check (Q1), i.e. to show that every stationary point is a strict local minimum.

⁹We take the normal cone definition by Clarke [14] for regions given by (3.10).

Figure 3.3 illustrates the normal cone definition and the boundary condition (B2). For a boundary point x , (B2) requires that $\nabla f(x)$ should not point in the opposite directions to the normal cone, i.e. the directions indicated by $-N_M(x)$ in the figure. In contrast, (B1) requires that $\nabla f(x)$ should point outward the region, which is a stronger condition as suggested by Figure 3.3. In the figure, D is a boundary point of M at which (f, M) satisfies both (B1) and (B2), (h, M) satisfies (B2) but not (B1), whereas the (k, M) satisfies neither.

We claim that (B2) is a weaker condition than (B1), i.e. that if (f, M) satisfies (B1), then it satisfies (B2). Assume the contrary, that (f, M) satisfies (B1) and not (B2). For $x \in \text{bd}(M)$ we have

$$-\nabla f(x) = \sum_{i \in I(x)} \lambda_i \nabla g_i(x)$$

for some $\lambda_i \geq 0$. We then have

$$-\sum_{i \in I(x)} \nabla f(x)^T \nabla g_i(x) = \sum_{i \in I(x)} \lambda_i \|\nabla g_i(x)\|^2.$$

The term on the right is nonnegative, but the term on the left is strictly negative since (f, M) satisfies (B1). This yields a contradiction and we conclude that (B2) is a weaker condition than (B1).

It is possible to generalize Theorem 3.3 and Proposition 3.2 to the case when (f, M) satisfies the weaker condition (B2). The constrained mountain pass theorems by Tintarev [57, 58, 59] and Schechter [52, 53] are of this nature, in particular, the result by Tintarev [59] could be used to generalize part (i) of Theorem 3.3 to hold when (f, M) satisfies (B2). We note that our proof of Theorem 3.3 can also be generalized so that part (ii) of Theorem 3.3 also holds when (f, M) satisfies (B2). We state the proposition and a sketch of the proof, and consider it a future research direction to make this argument rigorous.

Proposition 3.3 Let M be a non-empty, compact, and connected region given by (3.10). Let $U \subset \mathbb{R}^n$ be an open set containing M and $f : U \mapsto \mathbb{R}$ be a continuously

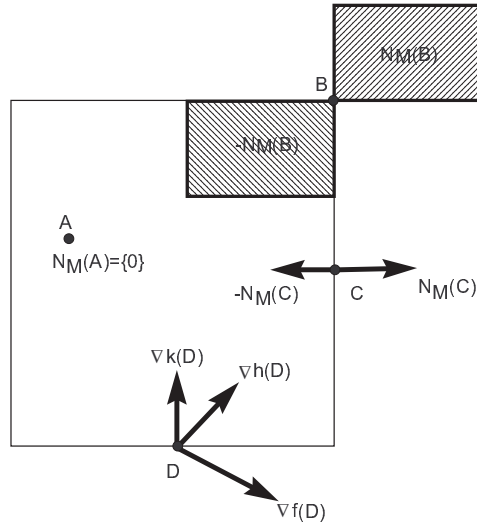


Figure 3.3: Illustration of the normal cone definition and the boundary condition (B2). (i) Consider 3 vectors A, B, C and the corresponding normal cones, as illustrated in the figure. Since A is an interior point, $N_M(A) = \{0\}$. Note that there are two [respectively, one] binding constraints at vector B [respectively, C], thus the normal cone is two [respectively, one] dimensional. (ii) The boundary condition (B2) requires that, given a vector x on the boundary, the gradient of the function at x should not point opposite the normal cone, i.e. directions indicated by $-N_M(x)$ in the figure. At point D , the functions f and h satisfy (B2) whereas the function k does not.

differentiable function. If (f, M) satisfies (Q1) and (B2), then (f, M) satisfies (P2).

Sketch of the Proof. We first show that Theorem 3.3 holds under the weaker assumption that (f, M) satisfies (B2). Consider a region M given by (3.10), an open set U containing M , and a continuously differentiable function $f : U \mapsto \mathbb{R}^n$. Assume that (f, M) satisfies (B2). In [59], Tintarev shows that it is then possible to find a pseudo-gradient vector field G for ∇f such that

$$G(x)' \nabla g_i(x) > 0, \quad \forall x \in M, i \in I(x).$$

Then, $G(x)$ is a proxy for the gradient which also satisfies the boundary assumption (B1) on the set M . One can then repeat the proof of Theorem 3.3 replacing ∇f with G , since the linear deformation along the pseudo-gradient (instead of the gradient) still strictly decreases the maximum function value of a connecting set [cf. proof of Theorem 3.3]. Once we extend Theorem 3.3 to hold under the weaker assumption (B2), Proposition 3.3 follows exactly the same way Proposition 3.2 follows from Theorem 3.3, completing the (sketch of the) proof.

Chapter 4

Topological Index Theory and the Uniqueness Property

In Example 2.4 of Section 2.2, we have seen that the number and properties of the stationary points of a function remain unchanged under differentiable deformations. In this chapter, we investigate the relationship between differential topology, in particular, the topological index theory, and the uniqueness problem. First, we consider examples in lower dimensions and observe patterns regarding the number and properties of the stationary points. We note that the patterns arise because the gradient vector field is constrained by its local properties and the topology of the region on which it is defined. We present the Poincare-Hopf Theorem which characterizes a relation satisfied by vector fields defined on smooth manifolds with boundary. We use the Poincare-Hopf Theorem to validate the previously observed patterns by establishing a relation satisfied by the properties of the function at its stationary points. We then present a generalization of the Poincare-Hopf Theorem due to Simsek, Ozdaglar, and Acemoglu [54], which allow us to generalize the previously established structural results regarding the stationary points to apply for the generalized stationary points (Karush-Kuhn-Tucker points). We finally use these results to establish sufficient conditions for the uniqueness of the local optimum under assumptions which treat the boundary and interior stationary points uniformly and which are, in that aspect, weaker than the assumptions of the uniqueness results of the previous chapter.

4.1 Global Constraints Satisfied by Local Properties: Observations in Lower Dimensions

We start with some preliminary definitions and results. The symmetric $n \times n$ matrix A is *positive semi-definite* if

$$x^T A x \geq 0, \quad \forall x \in \mathbb{R}^n. \quad (4.1)$$

The matrix A is *negative semi-definite* if

$$x^T A x \leq 0, \quad \forall x \in \mathbb{R}^n. \quad (4.2)$$

A is *positive definite* [respectively, *negative definite*] if the inequality in (4.1) [respectively, in (4.2)] is strict for all $x \neq 0$. The following proposition presents the well-known necessary and sufficient conditions for a vector to be a local minimum or a maximum of a twice continuously differentiable function (see Bertsekas [11] for the proof).

Proposition 4.1 Let $U \subset \mathbb{R}^n$ be an open set and $f : U \mapsto \mathbb{R}$ be a twice continuously differentiable function.

- (i) If $x \in U$ is a local minimum [respectively, local maximum] for f , then $\nabla f(x) = 0$ and $H_f(x)$ is positive semi-definite [respectively, negative semi-definite].
- (ii) If $\nabla f(x) = 0$ and $H_f(x)$ is positive definite [respectively, negative definite], then x is a strict local minimum [respectively, strict local maximum] of f .

Let $U \subset \mathbb{R}^n$ be an open set and $f : U \mapsto \mathbb{R}$ be a twice continuously differentiable function. A stationary point x of f is called *non-degenerate* if $H_f(x)$ is non-singular. If every stationary point of f is non-degenerate, then f is said to be *non-degenerate*. Intuitively, non-degenerate functions are not pathological and easier to analyze¹. We

¹From a transversality theory perspective, f is non-degenerate if the graph of ∇f intersects transversally with $U \times \{0\}$ in the ambient space $U \times \mathbb{R}^n$. Two sets that intersect transversally are

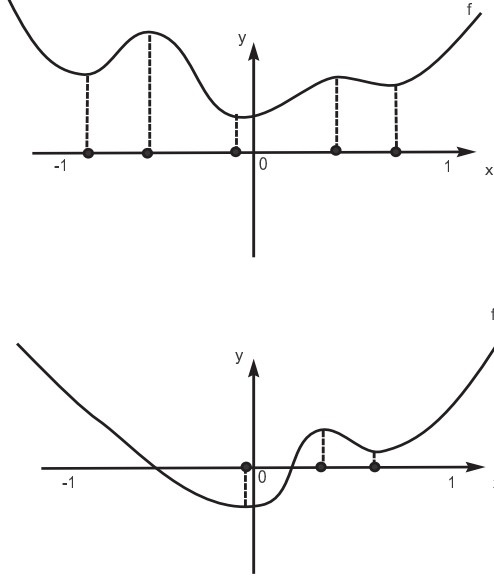


Figure 4.1: Illustration of 1-dimensional non-degenerate functions that satisfy (B1) on the region $\text{cl}(B(0, 1))$. In each case, note that f has alternating minima and maxima as one moves from left to right, and that the total number of minima is one greater than the total number of maxima.

analyze some special non-degenerate functions respectively in one and two dimensions.

Example 4.1 Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a non-degenerate twice continuously differentiable function. Assume that f satisfies the boundary condition (B1) for the region

$$\text{cl}(B(0, 1)) = \{x \in \mathbb{R} \mid x^2 \leq 1\},$$

i.e., that

$$\nabla f(x)^T x > 0, \quad \forall x \text{ such that } \|x\| = 1.$$

Figure 4.1 displays two such functions. Let $K = \{x \in \mathbb{R} \mid \nabla f(x) = 0\}$ be the set of

said to be in the *general position*. It can be shown that being in the general position is indeed a generic property, i.e. that it is satisfied by almost all set pairs. From here, it can be shown that non-degenerate functions are dense in the set of all continuously differentiable functions, making the concept of non-degeneracy a useful one (see Guillemin-Pollack [23] for an introduction to transversality theory).

the stationary points of f . Consider some $x \in K$. Since f is non-degenerate, there are two possibilities regarding the type of x .

- (i) $H_f(x) > 0$. In this case, by Proposition 4.1, x is a strict local minimum.
- (ii) $H_f(x) < 0$. This time, by Proposition, 4.1, f is a strict local maximum.

In other words, every stationary point of f is either a strict local minimum or a strict local maximum. We make the following observation from Figure 4.1.

Observation: The number of local minima of f is one more than the number of local maxima of f . Since the stationary points are characterized by the sign of the hessian, this is equivalent to

$$\sum_{x \in K} \text{sign}(H_f(x)) = 1 \quad (4.3)$$

where $\text{sign} : \mathbb{R} \mapsto \{-1, 0, 1\}$ is defined as

$$\text{sign}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Example 4.2 Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be a non-degenerate twice continuously differentiable function. Assume that f satisfies the boundary condition (B1) for the region

$$\text{cl}(B(0, 1)) = \{x \in \mathbb{R} \mid \|x\|^2 \leq 1\},$$

i.e., that

$$\nabla f(x)^T x > 0, \quad \forall x \text{ such that } \|x\| = 1.$$

Figure 4.2 demonstrates three such functions. Let $K = \{x \in \mathbb{R} \mid \nabla f(x) = 0\}$ be the set of the stationary points of f . Consider some vector $x \in K$. This time there are three possibilities for x .

- (i) $H_f(x)$ is positive-definite. Then, by Proposition 4.1, x is a strict local minimum. Since $H_f(x)$ has two eigenvalues both of which are positive and since the determinant

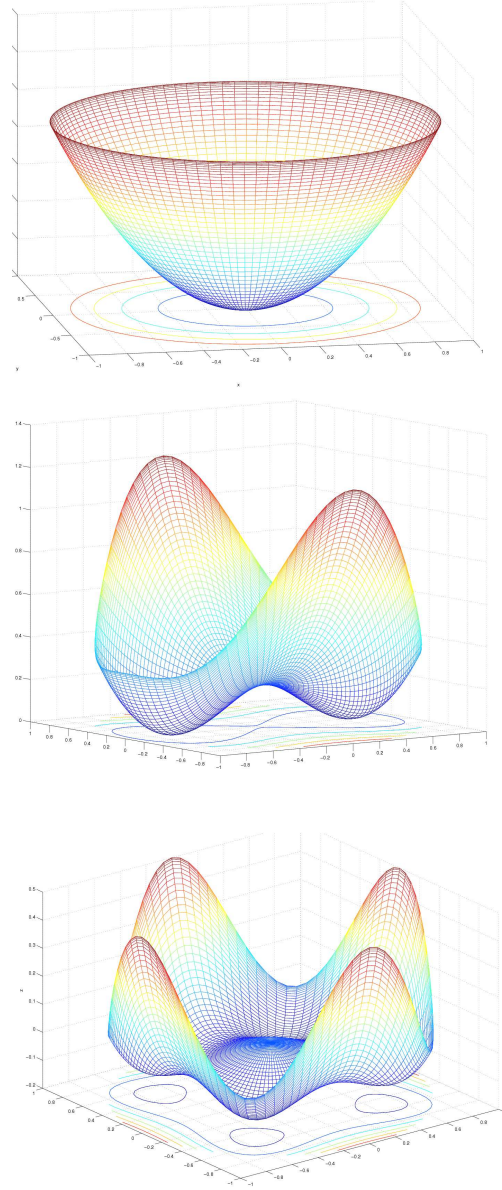


Figure 4.2: Illustration of 2-dimensional non-degenerate functions of Example 4.2. All of the above functions satisfy the boundary condition (B1) on the region $\text{cl}(B(0, 1))$. Note that the top function has a single stationary point, which is a strict local minimum. The middle function has three stationary points; two strict local minima and one saddle point. The bottom function has nine stationary points; four strict local minima, four saddle points, and one strict local maxima. Note that the relation, $\#(\text{local minima}) - \#(\text{saddle points}) + \#(\text{local maxima}) = 1$, holds in each case.

is the product of the eigenvalues, we have

$$\det(H_f(x)) > 0. \quad (4.4)$$

(ii) $H_f(x)$ is negative-definite. Then, by Proposition 4.1, x is a strict local maximum. Since $H_f(x)$ has two eigenvalues both of which are negative, we have

$$\det(H_f(x)) > 0. \quad (4.5)$$

(iii) $H_f(x)$ is non-singular, but neither positive-definite, nor negative-definite. From Proposition 4.1, x is neither a local minimum nor a local maximum, thus it is a saddle point. Moreover, $H_f(x)$ has two eigenvalues with opposite signs (since, otherwise, it would be either positive or negative definite). Then, we have

$$\det(H_f(x)) < 0. \quad (4.6)$$

From Figure 4.2, we observe the following pattern regarding such functions.

Observation: The sum of the number of strict local minima of f and the number of strict local maxima of f is one greater than the number of saddle points of f . From the characterization of the stationary points by Equations (4.4), (4.5), and (4.6), this is equivalent to

$$\sum_{x \in K} \text{sign}(\det(H_f(x))) = 1. \quad (4.7)$$

Observations we make in Examples 4.1 and 4.2 suggest that the local properties of f at its stationary points satisfy a global constraint. In the next section, we validate our observations by proving that the pattern observed in Equations (4.3) and (4.7) is a general property. We present the Poincare-Hopf Theorem, which is a deep result of differential topology that partially characterizes the behavior of any vector field on a given manifold. Considering the Poincare-Hopf Theorem for the special case when the vector valued function is the gradient, we obtain the relation in Eq. (4.7). We

then use this relation to obtain uniqueness results similar to Proposition 3.2.

4.2 Poincare-Hopf Theorem

Let M be a subset of the Euclidean set \mathbb{R}^n and $k \leq n$ be a positive integer. M is a *k-dimensional smooth manifold with boundary* if for each $x \in M$ there exists an open set $U \subset \mathbb{R}^n$ and a continuously differentiable function $f : U \mapsto \mathbb{R}^n$ with a continuously differentiable inverse such that f maps the set $U \cap (\mathbb{R}^{k-1} \times \mathbb{R}_+ \times \{0\}^{n-k})$ ² to a neighborhood of x on M ³. See Figure 4.3 for an illustration and examples.

Given an open set $U \subset \mathbb{R}^n$ and a continuously differentiable function $g : U \mapsto \mathbb{R}$, $c \in \mathbb{R}$ is a *regular value in the sense of Sard* if $\nabla g(x) \neq 0$ for all $x \in g^{-1}(\{c\})$. We will need the following lemma, which provides a simple sufficient condition for a region in \mathbb{R}^n to be an n -dimensional smooth manifold with boundary (see Milnor [35] for the proof).

Lemma 4.1 Let $U \subset \mathbb{R}^n$ be an open set, $g : U \mapsto \mathbb{R}$ be a differentiable function, and $c \in \mathbb{R}$ be a regular value of g such that $g^{-1}(\{c\})$ is not empty. Then

$$L_c = \{x \in U \mid g(x) \leq c\}$$

is an n -dimensional smooth manifold with boundary and

$$\text{bd}(L_c) = \{x \in U \mid g(x) = c\}.$$

The Poincare-Hopf Theorem relates the local properties of a vector field to the Euler characteristic of the underlying manifold, which is an integer associated with the topology of the manifold. For a compact set $M \subset \mathbb{R}^n$, we denote its *Euler charac-*

² \mathbb{R}_+ denotes the set of nonnegative real numbers.

³Our definition follows closely Mas-Colell [34] and Milnor [35], since we restrict ourselves to smooth manifolds embedded in an ambient Euclidean space \mathbb{R}^n . For a more abstract definition of smooth manifold (with or without boundary), see Guillemin-Pollack [23] or Hirsch [26].

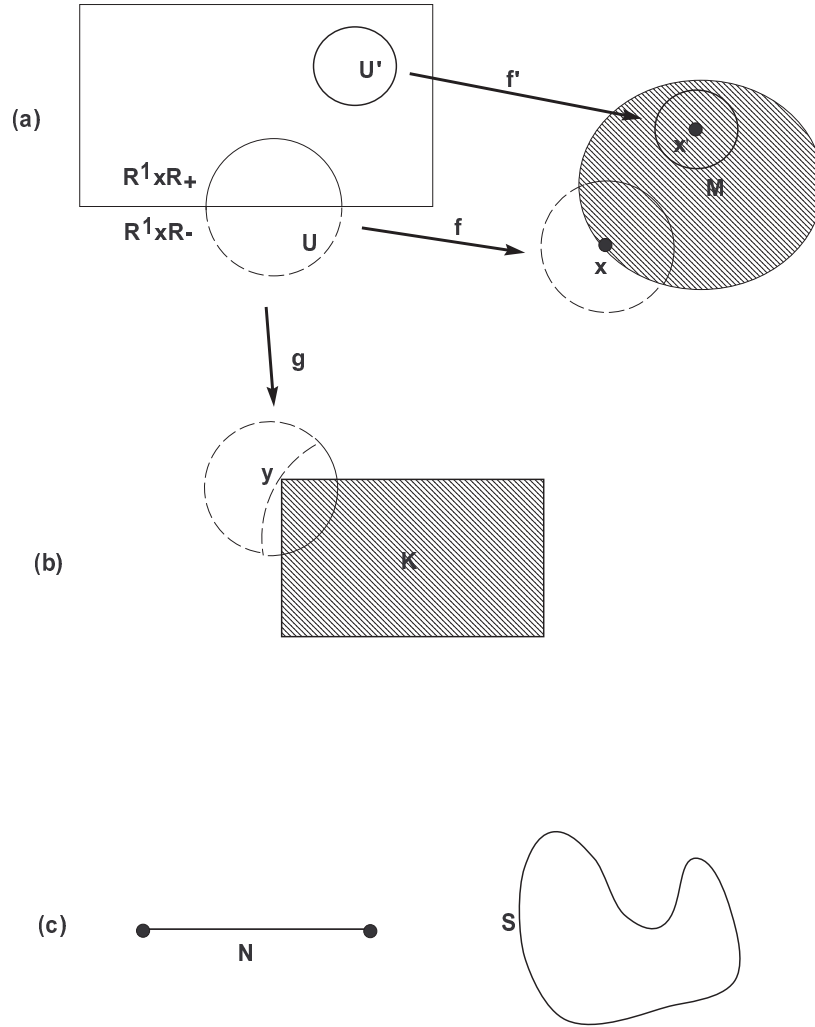


Figure 4.3: Illustration of the manifold definition for regions embedded in \mathbb{R}^2 . In (a), M is a smooth manifold with boundary, because given $x, x' \in M$, there exists functions f, f' which are differentiable with differentiable inverses, and which map the open subsets of the half space $\mathbb{R} \times \mathbb{R}_+$ exactly on sets that are open in M . In (b), K is not a smooth manifold with boundary, because for the corner point $y \in K$, no differentiable function (with differentiable inverse) maps the open subsets of the half space exactly on the open subsets of K . The example function g in the figure displays how this is not possible. In (c), N and S are 1-dimensional manifolds with boundary (the boundary of S is the empty set).

teristic with $\chi(M)$. Euler characteristic is a *topological invariant*, in fact, *homotopy invariant* of sets ⁴. We recall these notions in the following definition.

Definition 4.1 (a) Consider two sets $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$. M and N are *homeomorphic* if there exists a continuous function $f : M \mapsto N$ with continuous inverse $f^{-1} : N \mapsto M$.

(b) Consider two sets $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ homeomorphic to each other. A property is said to be a *topological invariant* if it holds for M if and only if it holds for N . Similarly, a set characteristic is a topological invariant if it is the same number for homeomorphic sets M and N .

(c) Let $M \subset \mathbb{R}^m$, $N \subset \mathbb{R}^n$ be sets. Consider two functions $f, g : M \mapsto N$. f and g are *homotopic* if there exists a continuous function $F : M \times [0, 1] \mapsto N$ such that

$$F(x, 0) = f(x) \quad \text{and} \quad F(x, 1) = g(x), \quad \text{for all } x \in M.$$

Such a function F is called a *homotopy* between f and g .

(d) Consider two sets $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$. M and N are said to be *homotopy equivalent* if there exists continuous functions $f : M \mapsto N$ and $g : N \mapsto M$ such that $f \circ g$ is homotopic to i_N and $g \circ f$ is homotopic to i_M , where $i_X : X \mapsto X$ denotes the identity function on some set X .

(e) Consider two sets $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ which are homotopy equivalent. A property is said to be a *homotopy invariant* if it holds for M if and only if it holds for N . Similarly, a set characteristic is a homotopy invariant if it is the same number for homotopy equivalent sets M and N .

Two sets $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ that are homotopy equivalent are also homeomorphic, but the converse implication is not true. Figure 4.2 displays some homotopy equivalent regions. The fact that homotopy equivalence of regions is rather easy to establish provides us with a practical way to calculate the Euler characteristic of a

⁴We refer to Hirsch [26], Guillemin-Pollack [23] for a definition and detailed discussion on the Euler characteristic of smooth manifolds, and to Rotman [49] for the Euler characteristic of more general simplicial complexes. See Massey [37] for a proof of the fact that Euler characteristic is a homotopy invariant.

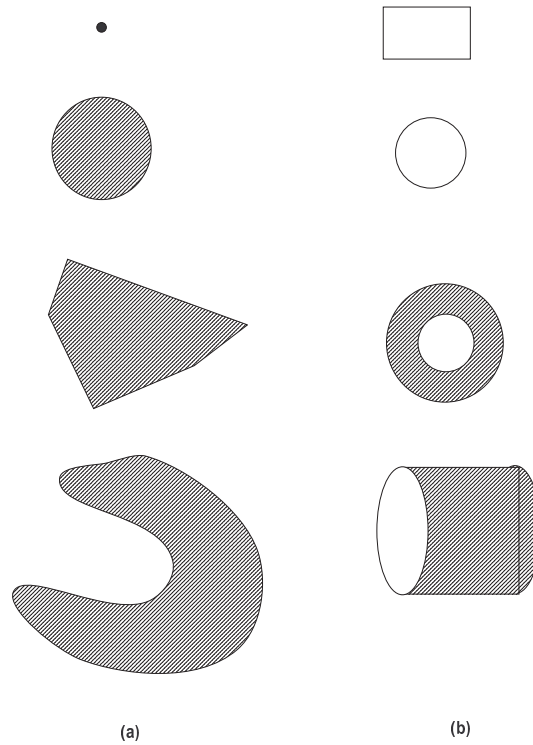


Figure 4.4: Illustration of homotopy equivalence. Intuitively, two sets are homotopy equivalent if one can continuously be deformed into the other. The sets in (a) are all homotopy equivalent to each other. They all have the same Euler characteristic, which is 1. Similarly, the sets in (b) are all homotopy equivalent to each other. Each of their Euler characteristic is 0. Note that, in *b*, the one dimensional circle, the two dimensional annulus, and the cylinder surface are all homotopic to each other, demonstrating that dimension of a set is not a homotopy invariant.

given region. We would try and find a region which is homotopic to the given region and whose Euler characteristic we already know. The following properties of the Euler characteristic will be sufficient to calculate the Euler characteristic of a rich set of regions.

1. Given non-empty, disjoint, compact sets $M, N \subset \mathbb{R}^n$,

$$\chi(M \cup N) = \chi(M) + \chi(N).$$

2. Given compact sets $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$,

$$\chi(M \times N) = \chi(M)\chi(N).$$

3. $\chi(B^n) = 1$ for any nonnegative integer n , where

$$B^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}.$$

4. $\chi(S^n) = 2$ for every nonnegative even integer n and $\chi(S^n) = 0$ for every nonnegative odd integer n , where

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}.$$

5. Given a non-empty, compact, convex set $M \subset \mathbb{R}^m$, $\chi(M) = 1$ ⁵.

We next define the Poincare-Hopf index of a zero of a vector field ⁶. Let $M \subset \mathbb{R}^n$ be an n -dimensional smooth manifold with boundary. Let U be an open set containing M and $F : U \mapsto \mathbb{R}^n$ be a continuous function. Let

$$Z(F, M) = \{x \in M \mid F(x) = 0\}$$

⁵One way to show this statement is to prove that every non-empty convex set $M \subset \mathbb{R}^n$ is homotopy equivalent to a single point, which has Euler characteristic 1 from property (i). See Borisovich et al. [7], Chapter 3 for details of this argument.

⁶In this chapter, we use the term *vector field* to represent a vector valued function from a subset of \mathbb{R}^n to \mathbb{R}^n . See Guillemin-Pollack [23] and Hirsch [26] for a definition of vector fields on general smooth manifolds.

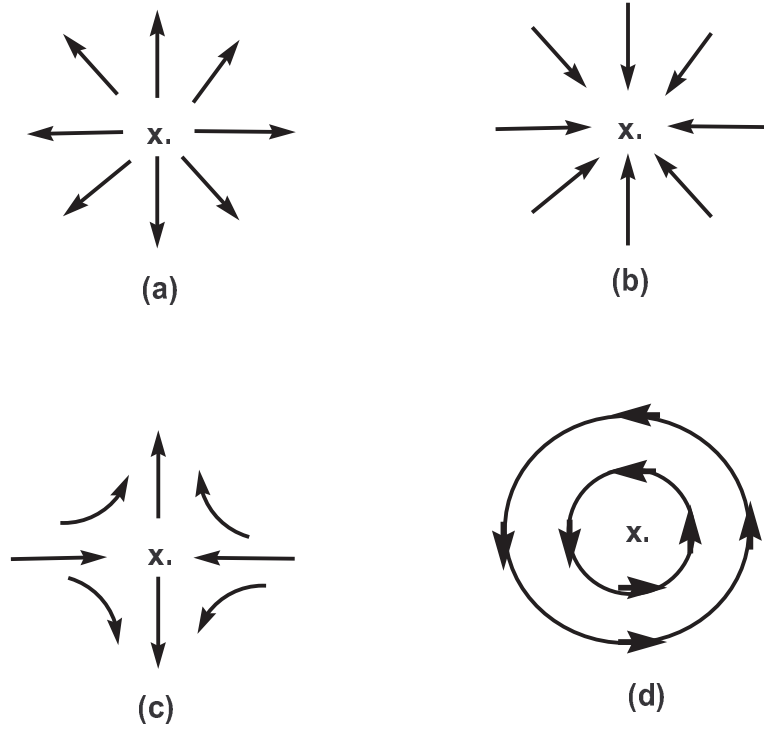


Figure 4.5: Illustration of Poincaré-Hopf indices of zeros of sample vector fields on the plane. The field in (a) is called a *source*. It has no real negative eigenvalues and the vector field flow is not moving toward x in any direction, so the index is $(-1)^0 = 1$. The field in (b) is called a *sink*. It has two real negative eigenvalues and the vector field flow is moving toward x in a subspace of dimension two, thus the index is $(-1)^2 = 1$. The field in (c) is called a *saddle*. It has one real negative eigenvalue and the vector field flow is moving toward x in a subspace of dimension 1, hence the index is $(-1)^1 = -1$. Finally, the field in (d) is called a *circulation*. Both of its eigenvalues are imaginary, it has no real negative eigenvalues and no directions at which the vector field flow is moving toward x , thus the index is $(-1)^0 = 1$.

denote the set of zeros of F over M . We say that $x \in Z$ is a *non-degenerate zero* of F if F is differentiable at x and $\nabla F(x)$ is non-singular. Given a non-degenerate zero $x \in Z(F, M)$, we define the *Poincare-Hopf index* of F at x as

$$\text{ind}(x) = \text{sign}(\det(\nabla F(x))).$$

Consider a non-degenerate zero $x \in Z$. Note that $\det(\nabla F(x))$ is equal to the product of the eigenvalues of $\nabla F(x)$. When $\nabla F(x)$ is a symmetric matrix, all of its eigenvalues are real and the index, $\text{sign}(\det(\nabla F(x)))$, is equal to $(-1)^k$ where k is the number of negative eigenvalues of f . Pictorially, every negative eigenvalue corresponds to a 1-dimensional space (eigenspace) along which F points toward x at nearby points, whereas every positive eigenvalue corresponds to a 1-dimensional space along which F points away from x . Then the index corresponds to $(-1)^k$ where k is the dimension of the largest subspace along which the vector field points toward x at nearby points (see Figure 4.5). When $\nabla(F(x))$ is not a symmetric matrix, then complex eigenvalues come in conjugate pairs, $(a + bi, a - bi)$, and the effect of each such eigenvalue on the sign of $\det(\nabla F(x))$ is cancelled by its conjugate since

$$(a + ib)^T(a - ib) = a^2 + b^2 > 0.$$

Then, once again, the index is $(-1)^k$ where k denotes the number of negative real eigenvalues of f . Pictorially, every complex eigenvalue pair represents a circulation (see Figure 4.5, (d)) for nearby points in a 2-dimensional space passing through x and negative [respectively, positive] eigenvalues correspond to directions at which F points toward [respectively, away from] x . Figure 4.5 illustrates the Poincare-Hopf indices of some vector valued functions on \mathbb{R}^2 .

The following is the celebrated Poincare-Hopf Theorem applied to vector fields on a smooth n -dimensional manifold $M \subset \mathbb{R}^n$ with boundary. The proof uses tools from differential topology and therefore is omitted (see Milnor [35], Guillemin-Pollack [23]). The intuition behind the proof is that the local properties of a vector field at its zeros are interrelated through differentiability properties of the vector field and the

underlying manifold, which puts a constraint on the possibilities for the number and the type of zeros the vector field can have. For example, if we consider a ball in \mathbb{R}^2 and try to draw vector field flows on it such that it points outward on the boundary, we realize that we are constrained in our drawing in terms of the number and indices of zeros we can have. Intuitively, following the flow of the vector field, we observe that whenever we leave a zero (contributing a $+$ to the index sum), we have to complete that flow line by entering another zero (contributing a $-$ to the index sum) or we have to leave the region (relating the indices to the topology of the region).

Poincare-Hopf Theorem

Let $M \subset \mathbb{R}^n$ be an n -dimensional compact smooth manifold with boundary. Let U be an open set containing M and $F : U \mapsto \mathbb{R}^n$ be a continuously differentiable function. Let

$$Z(F, M) = \{x \in M \mid F(x) = 0\}$$

denote the set of zeros of F over M . Assume the following:

- (A1) F points outward on the boundary of M . In other words given $x \in \text{bd}(M)$, there exists a sequence $\epsilon_i \downarrow 0$ such that $x + \epsilon_i F(x) \notin M$ for all $i \in \mathbb{Z}^+$.
- (A2) Every $x \in Z(F, M)$ is a non-degenerate zero of F .

Then, the sum of Poincare-Hopf indices corresponding to zeros of F over M equals the Euler characteristic of M . In other words,

$$\chi(M) = \sum_{x \in Z(F, M)} \text{sign}(\det(\nabla F(x))). \quad (4.8)$$

The theorem is also generalized to the case in which F is continuous and has isolated zeros, but is not necessarily differentiable (see Hirsch [26], Chapter 5 or Mas-Colell [34], Chapter 1). The index of $x \in Z$ in the continuous version is defined by means of approximating F with continuously differentiable functions, which can be shown to be the same as $\text{sign}(\det(F(x)))$ if F is differentiable at x ⁷. Therefore, we also have the following generalization of the Poincare-Hopf Theorem.

⁷See Hirsch [26] and Mas-Colell [34] for continuous generalizations of differential index theory.

Poincare-Hopf Theorem II

Let $M \subset \mathbb{R}^n$ be an n -dimensional compact smooth manifold with boundary. Let U be an open set containing M and $F : U \mapsto \mathbb{R}^n$ be a continuous function. Let $Z = \{x \in M \mid F(x) = 0\}$ denote the set of zeros of F over M . Assume the following:

(A1) F points outward on the boundary of M . In other words given $x \in \text{bd}(M)$, there exists a sequence $\epsilon_i \downarrow 0$ such that $x + \epsilon_i F(x) \notin M$ for all $i \in \mathbb{Z}^+$.

(A2) F is differentiable at every $x \in Z$.

(A3) Every $x \in Z(F, M)$ is a non-degenerate zero of F .

Then,

$$\sum_{x \in Z} \text{sign}(\det(F(x))) = \chi(M)$$

In the special case when the vector field in consideration is the gradient function, the Poincare-Hopf Theorem has implications on the properties of the stationary points. The following proposition formalizes the pattern observed in Equations (4.3) and (4.7) ⁸.

Proposition 4.2 Let M be a non-empty compact set given by (3.10) and $I = \{1\}$, i.e. let

$$M = \{x \mid g_1(x) \leq 0\}, \quad (4.9)$$

where $\nabla g_1(x) \neq 0$ for all $x \in M$ such that $g_1(x) = 0$ (i.e. g_1 satisfies the LICQ condition). Let $U \subset \mathbb{R}^n$ be an open set, and $f : U \mapsto \mathbb{R}$ be a non-degenerate twice continuously differentiable function. Assume that (f, M) satisfies (B1) and denote the set of the stationary points of f over M by

$$K(f, M) = \{x \in M \mid \nabla f(x) = 0\}.$$

⁸Note that Examples 4.1 and 4.2 are defined such that they satisfy (B1) for the unit ball, which has Euler characteristic equal to 1. Therefore, the sum of indices observed in the examples is equal to 1.

Then we have,

$$\chi(M) = \sum_{x \in K(f, M)} \text{sign}(\det(H_f(x))).$$

Proof. Note that M is a smooth manifold with boundary from Lemma 4.1. Given $x \in \text{bd}(M)$, the LICQ condition implies that $v \in \mathbb{R}^n$ is an outward direction if and only if

$$v^T \nabla g_1(x) > 0.$$

Then, since (f, M) satisfies (B1), $\nabla f(x)$ is an outward direction at every $x \in \text{bd}(M)$. Moreover,

$$Z(\nabla f, M) = K(f, M) \tag{4.10}$$

and since f is non-degenerate,

$$\nabla(\nabla f(x)) = H_f(x) \tag{4.11}$$

is non-singular at every $x \in Z(\nabla f, M)$. Then, the Poincare-Hopf Theorem applies for $F = \nabla f$ and using Equations (4.8), (4.10), and (4.11), we have

$$\begin{aligned} \chi(M) &= \sum_{x \in Z(\nabla f, M)} \text{sign}(\det(\nabla(\nabla f(x)))) \\ &= \sum_{x \in K(f, M)} \text{sign}(\det(H_f(x))) \end{aligned}$$

as desired. **Q.E.D.**

We immediately see that Proposition 4.2 has implications on the uniqueness problem. As one corollary, we obtain a result comparable to Proposition 3.2 of the previous chapter. Proposition 3.2 holds for regions given by finitely many smooth inequality constraints, which is not necessarily a smooth manifold. In contrast, this result is restricted to the case when the region is defined by a single smooth inequality constraint. We shall remedy this in the next section when we generalize the Poincare-Hopf Theorem to hold for regions given by finitely many inequality constraints. Also, note that

this proposition requires f to be non-degenerate and twice-continuously differentiable, whereas Proposition 3.2 requires f to be continuously differentiable. Note also that the topological requirements of the two results are not identical either. Proposition 3.2 requires the region to be connected, whereas this proposition requires it to have an Euler characteristic equal to 1.

Proposition 4.3 Let M be a non-empty compact set given by (4.9). Assume further that $\chi(M) = 1$. Let $U \subset \mathbb{R}^n$ be an open set containing M and $f : U \mapsto \mathbb{R}$ be a non-degenerate twice continuously differentiable function. If (f, M) satisfies (Q1) and (B1), then (f, M) satisfies (P2).

Proof. Consider a stationary point of f over M , i.e., consider a vector $x \in K(f, M)$. Since f satisfies (Q1), x is a strict local minimum. Then from Proposition 4.1, $H_f(x)$ is positive semi-definite. Since f is non-degenerate, $H_f(x)$ is also non-singular, and hence is positive definite. This implies that

$$\det(H_f(x)) > 0, \quad \forall x \in K(f, M).$$

Then by Proposition 4.2, we have

$$\chi(M) = \sum_{x \in K(f, M)} \text{sign}(\det(H_f(x))) = \sum_{x \in K(f, M)} 1. \quad (4.12)$$

Eq. (4.12) implies that the number of elements in $K(f, M)$ is equal to $\chi(M)$. Since $\chi(M) = 1$, $K(f, M)$ has a single element and f has a unique stationary point.

We claim that every local minimum x of f over M is a stationary point of f . Let x be a local minimum of f over M . From Karush-Kuhn-Tucker optimality conditions, $-\nabla f(x) \in N_M(x)$. Since (f, M) satisfies (B1), and thus the weaker condition (B2), it must be the case that $x \notin \text{bd}(M)$. Then $x \in \text{int}(M)$, $N_M(x) = \{0\}$, and consequently $\nabla f(x) = 0$, which implies x is a stationary point.

Since every local minimum of f over M is a stationary point of f and f has a unique stationary point, we conclude that f has at most one (indeed exactly one)

local minimum. f satisfies (P2) as desired. **Q.E.D.**

Propositions 4.2 and 4.3 validate our observations of Section 4.1 and suggest that topological index theory is useful in analyzing the uniqueness problem. In the next section, we present our generalized Poincare-Hopf Theorem to further exploit this connection.

4.3 A Generalized Poincare-Hopf Theorem

In Simsek-Ozdaglar-Acemoglu, we generalize the Poincare-Hopf Theorem applied to n -dimensional manifolds with boundary in \mathbb{R}^n in a number of ways. First, we relax the smooth manifold assumption and let M be a region defined by a finite number of smooth inequality constraints which is not necessarily a smooth manifold. Second, we generalize the notion of critical points to include critical points on the boundary. In this section we present our result and the proof. We start with some preliminary definitions.

4.3.1 Preliminary Definitions and the Main Result

Let M be given by (3.10), i.e. let

$$M = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, \ i \in I = \{1, 2, \dots, |I|\}\}, \quad (4.13)$$

where the $g_i : \mathbb{R}^n \mapsto \mathbb{R}$, $i \in I$, are twice continuously differentiable. For some $x \in M$, let $I(x) = \{i \in I \mid g_i(x) = 0\}$ denote the set of active constraints. Assume that every $x \in M$ satisfies the LICQ condition [cf. Section 3.2]. For $x \in M$, we define the $n \times |I(x)|$ matrix

$$G(x) = [\nabla g_i(x)|_{i \in I(x)}], \quad (4.14)$$

where columns ∇g_i are ordered in increasing order of i . Furthermore, we complete the vectors $\{g_i(x)|i \in I(x)\}$ to a basis of \mathbb{R}^n with arbitrary but fixed $\{v_j, j \in I^c(x) = \{|I(x)| + 1, \dots, n\}\}$ such that $g_i(x)^T v_j = 0$ for all i, j and v_j 's are orthonormal. We

denote $V(x) = [v_j]_{j \in I^c(x)}$ and $C(x) = [G(x) \ V(x)]$. We call the completed basis a *normal-tangent basis* for $x \in M$. Note that $C(x)$ is a change of coordinates matrix from normal-tangent basis coordinates to standard coordinates. Then the normal cone at $x \in M$ can be written as

$$N_M(x) = \{v \in \mathbb{R}^n \mid v = G(x)\lambda, \lambda \in \mathbb{R}^{I(x)}, \lambda \geq 0\}.$$

We define the *boundary of the normal cone of M at x* , $\text{bd}(N_M(x))$, by

$$\text{bd}(N_M(x)) = N_M(x) - \text{ri}(N_M(x)),$$

where $\text{ri}(N_M(x))$ is the relative interior of the convex set $N_M(x)$, i.e.,

$$\text{ri}(N_M(x)) = \{v \in \mathbb{R}^n \mid v = G(x)\lambda, \lambda \in \mathbb{R}^{I(x)}, \lambda > 0\}.$$

If $I(x) = \emptyset$, we define $N_M(x) = \{0\}$ and $\text{bd}(N_M(x)) = \emptyset$.

Our extension of the Poincare-Hopf Theorem applies for a generalized notion of critical points of a function on M .

Definition 4.2 Let M be a region given by (4.13). Let U be an open set containing M and $F : U \mapsto \mathbb{R}^n$ be a continuously differentiable function.

- (a) We say that $x \in M$ is a *generalized critical point of F over M* if $-F(x) \in N_M(x)$.
- ⁹. We denote the set of generalized critical points of F over M by $\text{Cr}(F, M)$.
- (b) For $x \in \text{Cr}(F, M)$, we define $\theta(x) \geq 0$ to be the unique vector in $\mathbb{R}^{|I(x)|}$ which satisfies

$$F(x) + G(x)\theta(x) = 0.$$

We say that $x \in M$ is a *complementary critical point* if $-F(x) \in \text{ri}(N_M(x))$. In other

⁹Given a correspondence $F : M \rightrightarrows \mathbb{R}^n$, Cornet [17] calls $x \in M$ a generalized equilibrium of F if $F(x) \cap N_M(x) \neq \emptyset$. Ours could be considered a special case of that definition when F is single valued, motivating the term *generalized critical point*. However, our definition is not exactly the same as Cornet's *generalized equilibrium* definition since we characterize the critical points with $-F(x) \in N_M(x)$ rather than $F(x) \in N_M(x)$. This yields notational convenience in the generalized index theory we subsequently develop.

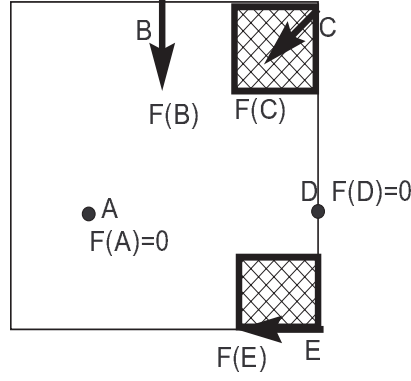


Figure 4.6: Illustration of critical points. Note that A, B, C are complementary critical points, whereas D and E are critical points that are not complementary.

words, $x \in \text{Cr}(F, M)$ is complementary if and only if $\theta(x) > 0$.

(c) We define

$$\Gamma(x) = V(x)^T \left(\nabla F(x) + \sum_{i \in I(x)} \theta_i(x) H_{g_i}(x) \right) V(x). \quad (4.15)$$

We say that x is a *non-degenerate critical point* if $\Gamma(x)$ is a non-singular matrix.

For an optimization problem, the local optima (minima or maxima) are the generalized critical points of the gradient mapping of the objective function. To see this, let M be the region given by (4.13), U be an open set containing M , and $f : U \mapsto \mathbb{R}$ be a twice continuously differentiable function. Consider the following optimization problem,

$$\begin{aligned} & \min f(x) \\ & \text{subject to } x \in M. \end{aligned} \quad (4.16)$$

Let x^* be a local optimum of (4.16). If $x^* \in \text{int}(M)$, then by the unconstrained optimality conditions, we have $\nabla f(x^*) = 0$. Since $N_M(x^*) = \{0\}$, we have $-\nabla f(x^*) \in N_M(x^*)$ and x^* is a generalized critical point of ∇f over M . For general (not necessarily interior) x^* , we have $-\nabla f(x^*) \in N_M(x^*)$, which is the optimality condition

for optimization over an abstract set constraint (see Bertsekas-Nedic-Ozdaglar [12]). Therefore, every local optima of f is a generalized critical point of ∇f over M .

We now define the notion of the index of a critical point and state our result.

Definition 4.3 Let M be a region given by (4.13). Let U be an open set containing M and $F : U \mapsto \mathbb{R}^n$ be a continuously differentiable function. Let $x \in M$ be a complementary and non-degenerate critical point of F over M . We define the index of F at x as

$$\text{ind}_F(x) = \text{sign}(\det(\Gamma(x))).$$

Note that the definition of $\text{ind}_F(x)$ is independent of the choice of $V(x)$. A change from a tangent basis $V(x)$ to another basis $V'(x)$ can be viewed as a change of coordinates over the tangent space¹⁰. Since $\Gamma(x)$ is a linear operator from the tangent space to itself, a change of coordinates does not change its determinant [cf. Eq. (4.15)].

Theorem 4.1 Let M be a region given by (4.13). Let U be an open set containing M and $F : U \mapsto \mathbb{R}^n$ be a continuously differentiable function. Assume that every critical point of F over M is complementary and non-degenerate¹¹. Then, the following generalization of the Poincare-Hopf Theorem to generalized critical points holds on the region M :

$$\chi(M) = \sum_{x \in \text{Cr}(F, M)} \text{ind}_F(x).$$

Note that, when the vector field F points outward on the region M , then all generalized critical points of F are interior points of M , which by definition 4.2,

¹⁰We call the space spanned by the columns of $G(x)$ the *normal space* at x . The *tangent space* is, then, the space perpendicular to the normal space. Note that, by choice of $V(x)$, the columns of $V(x)$ constitute a basis for the tangent space.

¹¹The complementarity condition corresponds to the strict complementary slackness condition in optimization theory. The non-degeneracy condition is the extension of assumption (A2) in the Poincare-Hopf Theorem II.

implies that they are zeros of F . Thus we obtain the Poincare-Hopf Theorem as a special case of Theorem 4.1. Theorem 4.1 removes the assumption that requires the vector field to point outward on the boundary and allows for critical points on the boundary by also accounting for their contribution to the index sum, where, in contrast, the original Poincare-Hopf Theorem restricts the critical points to be in the interior by means of a boundary condition.

We present the proof of Theorem 4.1 in the remainder of this section. The intermediary results shown for the proof are interesting on their own, but are not directly related to the uniqueness problem, hence the reader who is interested in the uniqueness results could skip the rest of the section and go to section 4.4.

We will prove Theorem 4.1 using an extension theorem which we develop in the next subsection and which is of independent interest.

4.3.2 Local Extension to a Smooth Manifold

Given $\epsilon > 0$, let

$$M^\epsilon = \{x \in \mathbb{R}^n \mid \|x - y\| < \epsilon \text{ for all } y \in M\}.$$

In other words, M^ϵ denotes the set of points with distance to M strictly less than ϵ . Note that M^ϵ is an open set.

Our goal is to extend F to M^ϵ in a way that maps the generalized critical points of F to regular critical points of the extended function. Our extension relies on the properties of the projection function in a neighborhood of M .

4.3.2.1 Properties of the Euclidean Projection

In this subsection, we define the projection of a vector x in \mathbb{R}^n on a closed possibly nonconvex set and explore its properties. We show that projection of proximal points on a nonconvex set inherits most of the properties of projection on a convex set.

Definition 4.4 We define the projection correspondence $\pi : \mathbb{R}^n \mapsto M$ as

$$\pi(y) = \arg \min_{x \in M} \|y - x\|.$$

We also define the distance function $d : \mathbb{R}^n \mapsto \mathbb{R}$ as

$$d(y) = \inf_{x \in M} \|y - x\|.$$

We note from Berge's maximum theorem [8] that π is upper semi-continuous and d is continuous. Also since M is closed, we have $x \in M$ if and only if $d(x) = 0$. We can then characterize the sets M and M^ϵ as

$$M = \{x \in \mathbb{R}^n | d(x) = 0\},$$

and

$$M^\epsilon = \{x \in \mathbb{R}^n | d(x) < \epsilon\}.$$

We next show that for sufficiently small ϵ , the projection correspondence $\pi|_{M^\epsilon}$ is single-valued and Lipschitz continuous.

Proposition 4.4 There exists $\epsilon > 0$ such that $\pi|_{M^\epsilon}$ is a globally Lipschitz function over M^ϵ . In other words, $\pi(x)$ is single-valued for all $x \in M^\epsilon$ and there exists $k > 0$ such that

$$\|\pi(x) - \pi(y)\| \leq k\|x - y\|, \quad \forall x, y \in M^\epsilon.$$

We need some preliminary results to prove Proposition 4.4. We first note the following lemma which is a direct consequence of the LICQ condition.

Lemma 4.2 There exists some scalar $m > 0$ such that for all $x \in M$ and $\lambda \in \mathbb{R}^{|I(x)|}$,

we have

$$\|G(X)\lambda\| \geq m \sum_{i \in I(x)} |\lambda_i|, \quad (4.17)$$

[cf. Eq. (4.14)].

Proof. Let $\mathcal{P}(I)$ denote the set of all subsets of I . Note that $I(x) \in \mathcal{P}(I)$ for all $x \in M$. For $S \in \mathcal{P}(I)$ define,

$$G_S = \{x \in M \mid g_i(x) = 0 \text{ if } x \in S\}$$

Clearly, G_S is closed and thus compact. By the LICQ condition, for every $x \in G_S$, the vectors $[\nabla g_i(x)]_{i \in S}$ are linearly independent. This implies that the following minimization problem has a positive solution:

$$m_S(x) = \min_{\|u\|=1, u \in \mathbb{R}^{|S|}} \left\| \sum_{i \in S} u_i \nabla g_i(x) \right\|.$$

By Berge's maximum theorem [8], $m_S(x)$ is continuous in x . Then, $m_S(x) > 0$ for all $x \in G_S$ implies there exists $m_S > 0$ such that $m_S(x) \geq m_S$ for all $x \in G_S$. Since the set $\mathcal{P}(I)$ is finite, $m' = \min_{S \in \mathcal{P}(I)} m_S > 0$. Then, for any $x \in M$ and $\lambda \in \mathbb{R}^{|I(x)|}$ and $\lambda \neq 0$, it follows that

$$\left\| \sum_{i \in I(x)} \frac{\lambda_i}{\|\lambda\|} \nabla g_i(x) \right\| \geq m',$$

and thus

$$\begin{aligned} \left\| \sum_{i \in I(x)} \lambda_i \nabla g_i(x) \right\| &\geq m' \|\lambda\| \\ &\geq \frac{m'}{|I(x)|} \sum_{i \in I(x)} |\lambda_i| \\ &\geq \frac{m'}{|I|} \sum_{i \in I(x)} |\lambda_i| \end{aligned}$$

where to get the second inequality, we used the arithmetic mean-geometric mean

inequality (see [12]). Since the inequality is also true for $\lambda = 0$, we conclude that the result holds with $m = m'/|I|$. **Q.E.D.**

Definition 4.5 Given $\epsilon > 0$, we define the ϵ -normal correspondence $N^\epsilon : M \mapsto R^n$ as

$$N^\epsilon(x) = (x + N_M(x)) \cap M^\epsilon.$$

We also define the correspondence $\text{ri}(N^\epsilon(x))$ as

$$\text{ri}(N^\epsilon(x)) = (x + B(x, \epsilon)) \cap \text{ri}(N_M(x)).$$

The following lemma shows that for sufficiently small positive ϵ , the ϵ -normal correspondence satisfies a Lipschitzian property.

Lemma 4.3 There exists $\epsilon > 0$ and $k > 0$ such that for all $x, y \in M$, and $s_x \in N^\epsilon(x)$, $s_y \in N^\epsilon(y)$, we have

$$\|y - x\| \leq k\|s_y - s_x\|.$$

Proof. For some $i \in I$, we define the function $e_i : U \times U \mapsto \mathbb{R}$ as

$$e_i(x, y) = \begin{cases} \frac{g_i(y) - g_i(x) + \nabla g_i(x)^T (y - x) + 1/2 (y - x)^T H_{g_i}(x) (y - x)}{\|y - x\|^2} & \text{if } y \neq x \\ 0 & \text{if } y = x, \end{cases} \quad (4.18)$$

Since g_i is twice continuously differentiable, the function e_i is continuous. Thus, $|e_i|$ has a maximum over the compact set $M \times M$, i.e., there exists some $\mu > 0$ such that

$$\|e_i(x, y)\| < \mu, \quad \forall x, y \in M. \quad (4.19)$$

Let $H_i = \max_{x \in M} \|H_{g_i}(x)\|$ ¹². Let $H_m = \max_{i \in I} H_i$. Also, let $m > 0$ be a scalar that satisfies Eq. (4.17) in Lemma 4.2.

¹²Note that the maximum exists since $\|H_{g_i}\|$ is a continuous function over the compact region M

We will prove that the result in Lemma 4.3 holds for

$$\epsilon = \frac{m}{2n(H_m + 2\mu)} > 0, \quad (4.20)$$

and $k = 2$. Let $x, y \in M$ and s_x, s_y in $N^\epsilon(x)$ and $N^\epsilon(y)$ respectively. If $y = x$ then we are done. Assume $y \neq x$. Then, by the definition of the ϵ -normal correspondence and the normal cone, there exist scalars λ_i , $i \in I(x)$ and $\gamma_j \geq 0$, $j \in I(y)$ such that

$$s_x = x + \sum_{i \in I(x)} \lambda_i \nabla g_i(x), \quad (4.21)$$

$$s_y = y + \sum_{j \in I(y)} \gamma_j \nabla g_j(y). \quad (4.22)$$

Using Lemma 4.2 and the fact that $s_x \in N^\epsilon(x)$, we obtain

$$m \sum_{i \in I(x)} |\lambda_i| \leq \left\| \sum_{i \in I(x)} \lambda_i \nabla g_i(x) \right\| \leq \epsilon = \frac{m}{2(H_m + 2\mu)},$$

implying that

$$\sum_{i \in I(x)} |\lambda_i| \leq \frac{1}{2(H_m + 2\mu)}.$$

Similarly,

$$\sum_{j \in I(y)} |\gamma_j| \leq \frac{1}{2(H_m + 2\mu)}.$$

Using the definition of the function e_i [cf. Eq. (4.18)], we have for all i ,

$$g_i(y) = g_i(x) + \nabla g_i(x)^T (y - x) + \frac{1}{2} (y - x)^T H_{g_i}(x) (y - x) + \epsilon_i(y, x) \|y - x\|^2$$

and,

$$g_i(x) = g_i(y) + \nabla g_i(y)^T (x - y) + \frac{1}{2} (x - y)^T H_{g_i}(y) (x - y) + \epsilon_i(x, y) \|x - y\|^2$$

Multiplying the preceding relations with λ_i and γ_j respectively, and summing over

$i \in I(x)$ and $j \in I(y)$, we obtain

$$\begin{aligned}
\sum_{i \in I(x)} \lambda_i g_i(y) + \sum_{j \in I(y)} \gamma_j g_j(x) &= \left(\sum_{i \in I(x)} \lambda_i \nabla g_i(x) - \sum_{j \in I(y)} \gamma_j \nabla g_j(y) \right)^T (y - x) \\
&+ \sum_{i \in I(x)} \lambda_i \left(\frac{1}{2} (y - x)^T H_{g_i}(x) (y - x) + \epsilon_i(y, x) \|y - x\|^2 \right) \\
&+ \sum_{j \in I(y)} \gamma_j \left(\frac{1}{2} (x - y)^T H_{g_j}(y) (x - y) + \epsilon_j(x, y) \|x - y\|^2 \right)
\end{aligned}$$

Since $x, y \in M$ and $\lambda_i \geq 0$, $\gamma_j \geq 0$, the term on the left hand side Eq. (4.23) is non-positive. By Equations (4.21) and (4.22), it follows that the first term on the right hand side is equal to

$$(s_x - x - s_y + y)^T (y - x) = (s_x - s_y)^T (y - x) + \|y - x\|^2.$$

Combining Eq. (4.19) with the bound on the norm of the hessian, it can be seen that the second term on the right is bounded below by

$$\sum_{i \in I(x)} -|\lambda_i| (H_m/2 + \mu) \|y - x\|^2 \geq -\frac{1}{2(H_m + 2\mu)} |H_m/2 + \mu| \|y - x\|^2 \geq -1/4 \|y - x\|^2.$$

Similarly, the last term on the right hand side is bounded below by $-1/4 \|y - x\|^2$.

Combining the above relations, Eq. (4.23) yields

$$0 \geq (s_x - s_y)^T (y - x) + \|y - x\|^2 - \frac{1}{4} \|y - x\|^2 - \frac{1}{4} \|y - x\|^2,$$

which implies that

$$-\frac{1}{2} \|y - x\|^2 \geq (s_x - s_y)^T (y - x) \geq -\|s_y - s_x\| \|y - x\|,$$

where we used the Cauchy-Schwarz inequality to get the second inequality. Finally, since $y - x \neq 0$, we obtain

$$\|y - x\| \leq 2\|s_y - s_x\|.$$

Hence, the claim is satisfied with the ϵ given by Eq. (4.20) and $k = 2$. **Q.E.D.**

For the rest of this subsection, let $\epsilon > 0$, $k > 0$ be fixed scalars that satisfy the claim of Lemma 4.3. The following is a corollary of Lemma 4.3 and shows that the ϵ -normal correspondence is injective.

Corollary 4.1 Given $x, y \in M$, if $x \neq y$, then $N^\epsilon(x) \cap N^\epsilon(y) = \emptyset$.

Proof. Let $v \in N^\epsilon(x) \cap N^\epsilon(y)$. Then, by Lemma 4.3, $\|x - y\| \leq k\|v - v\| = 0$ implies that $\|x - y\| = 0$ and hence $x = y$ as desired.

We next note the following lemma which shows that the correspondence N^ϵ is the inverse image of the projection correspondence.

Lemma 4.4 Given $y \in M^\epsilon$ and $p \in M$, $p \in \pi(y)$ if and only if $y \in N^\epsilon(p)$.

Proof. Let $y \in M^\epsilon$ and $p \in \pi(y)$. Then, by the optimality conditions, we have $y - p \in N_M(p)$ and thus $y \in N^\epsilon(p)$. Conversely, let $y \in N^\epsilon(p)$ for some $p \in M$. Assume that $p \notin \pi(y)$. Then there exists $p' \in \pi(y)$ such that $p' \neq p$. Then $y \in N^\epsilon(p') \cap N^\epsilon(p)$, which is a contradiction by Lemma 4.1. Therefore, we must have $p \in \pi(y)$, completing the proof.

Proof of Proposition 4.4. Assume that there is some $x \in M^\epsilon$ such that $\pi(x)$ is not single-valued. Then, there exist $p, q \in \pi(x) \subset M$ such that $p \neq q$. By Lemma 4.4, $x \in N^\epsilon(p)$ and $x \in N^\epsilon(q)$, therefore $x \in N^\epsilon(p) \cap N^\epsilon(q)$, contradicting Lemma 4.1. Therefore, we conclude that $\pi|_{M^\epsilon}$ is single-valued.

Let $x, y \in M^\epsilon$. Then, $x \in N^\epsilon(\pi(x))$ and $y \in N^\epsilon(\pi(y))$ [cf. Lemma 4.4], and it follows by Lemma 4.3 that

$$\|\pi(x) - \pi(y)\| \leq k\|x - y\|,$$

showing that π is globally Lipschitz. **Q.E.D.**

The next proposition shows that the distance function restricted to $M^\epsilon - M$,

$d|_{M^\epsilon - M}$, is continuously differentiable.

Proposition 4.5 The distance function d is continuously differentiable for all $x \in M^\epsilon - M$ with derivative

$$\nabla d(x) = \frac{x - \pi(x)}{d(x)}.$$

For the proof, we need the following result.

Lemma 4.5 For any $x \in M^\epsilon$ and $z \in M$, we have

$$(x - z)^T(\pi(x) - z) \geq 0.$$

Proof. For any $x \in M^\epsilon$ and $z \in M$, we can write $x - \pi(x) = (x - z) + (z - \pi(x))$, which implies that

$$\|x - \pi(x)\|^2 = \|x - z\|^2 + 2(x - z)^T(z - \pi(x)) + \|z - \pi(x)\|^2.$$

By the definition of $\pi(x)$, we have $\|x - \pi(x)\| \leq \|x - z\|$, therefore the preceding implies that $(x - z)^T(\pi(x) - z) \geq 0$, completing the proof. **Q.E.D.**

We next note the following lemma regarding the differentiability properties of functions that have values close to each other.

Lemma 4.6 Let $A \subset \mathbb{R}^n$ be an open set, and $f, g : A \mapsto \mathbb{R}$ be scalar valued functions such that f is differentiable at $x \in A$. Assume that $f(x) = g(x)$ and there exists $K > 0$ such that

$$\|f(y) - g(y)\| \leq K\|y - x\|^2, \quad \forall y \in A. \quad (4.23)$$

Then, g is differentiable at x with derivative equal to $\nabla f(x)$.

Proof. Define

$$e_x^f(v) = \frac{f(x+v) - f(x) - \nabla f(x)^T v}{\|v\|}.$$

The assumption that f is differentiable at $x \in A$ implies that

$$\lim_{v \rightarrow 0} e_x^f(v) = 0.$$

By (4.23), for all $v \neq 0$ and sufficiently small such that $x+v \in A$, we have

$$e_x^g(v) = \frac{g(x+v) - g(x) - \nabla f(x)^T v}{\|v\|} \leq \frac{f(x+v) + K\|v\|^2 - f(x) - \nabla f(x)^T v}{\|v\|} = e_x^f(v) + K\|v\|.$$

Similarly,

$$e_x^g(v) \geq \frac{f(x+v) - K\|v\|^2 - f(x) - \nabla f(x)^T v}{\|v\|} = e_x^f(v) - K\|v\|.$$

Combining the preceding two relations, we obtain

$$e_x^f(v) - K\|v\| \leq e_x^g(v) \leq e_x^f(v) + K\|v\|.$$

By taking the limit as $v \rightarrow 0$, this yields

$$0 = \lim_{v \rightarrow 0} e_x^f(v) - K\|v\| \leq \lim_{v \rightarrow 0} e_x^g(v) \leq \lim_{v \rightarrow 0} e_x^f(v) + K\|v\| = 0.$$

Thus, $\lim_{v \rightarrow 0} e_x^g(v) = 0$, showing that g is differentiable at x with derivative $\nabla f(x)$ as desired. **Q.E.D.**

Proof of Proposition 4.5. Let x be an arbitrary vector in $M^\epsilon - M$. Consider the function $f : M^\epsilon - M \mapsto \mathbb{R}$ given by

$$f(w) = \|w - \pi(x)\|.$$

Let $\delta \in R$ be such that $0 < \delta < d(x)$. Then, f is differentiable on the ball $B(x, \delta)$

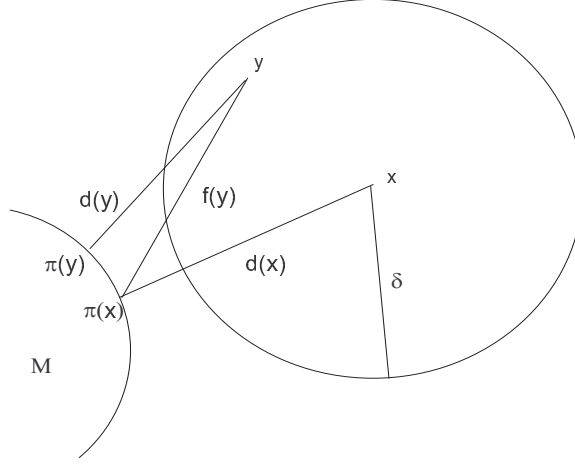


Figure 4.7: The distance function d is close to the function f in a neighborhood of x . Then since f is differentiable at x , d is also differentiable at x with derivative $\nabla f(x)$.

with derivative

$$\nabla f(w) = \frac{w - \pi(x)}{\|w - \pi(x)\|}.$$

Let $y \in B(x, \delta)$. By the definition of $d(y)$ and δ , we have

$$f(y) \geq d(y) \geq d(x) - \delta > 0. \quad (4.24)$$

Using $y - \pi(x) = y - \pi(y) + (\pi(y) - \pi(x))$, we obtain

$$\begin{aligned} f(y)^2 = \|y - \pi(x)\|^2 &= \|y - \pi(y)\|^2 + \|\pi(y) - \pi(x)\|^2 + 2(y - \pi(y))^T(\pi(y) - \pi(x)) \\ &= d(y)^2 + \|\pi(y) - \pi(x)\|^2 + 2(y - x + x - \pi(y))^T(\pi(y) - \pi(x)) \\ &= d(y)^2 + \|\pi(y) - \pi(x)\|^2 + 2(y - x)^T(\pi(y) - \pi(x)) \\ &\quad + 2(x - \pi(y))^T(\pi(y) - \pi(x)) \\ &\leq d(y)^2 + \|\pi(y) - \pi(x)\|^2 + 2\|y - x\|\|\pi(y) - \pi(x)\| \\ &\leq d(y)^2 + (k^2 + 2k)\|y - x\|^2 \end{aligned}$$

where we used Lemma 4.5 [with $z = \pi(y)$] and the Cauchy-Schwarz inequality to get the first inequality and the fact that π is globally Lipschitz over M^ϵ [cf. Proposition 4.4] to get the second inequality. Using the preceding, we obtain

$$f(y)^2 - d(y)^2 \leq (k^2 + 2k)\|y - x\|^2,$$

which implies

$$f(y) - d(y) \leq \frac{k^2 + 2k}{f(y) + d(y)}\|y - x\|^2.$$

Using Eq. (4.24), we further obtain

$$f(y) - d(y) \leq \frac{k^2 + 2k}{2(d(x) - \delta)}\|y - x\|^2.$$

Then, for

$$K = \frac{k^2 + 2k}{2(d(x) - \delta)} > 0,$$

we have

$$0 \leq f(y) - d(y) \leq K\|y - x\|^2$$

for all $y \in B(x, \delta)$. Since $f(x) = d(x)$, we conclude by Lemma 4.6 that d is differentiable at x with derivative

$$\nabla d(x) = \nabla f(x) = \frac{x - \pi(x)}{\|x - \pi(x)\|} = \frac{x - \pi(x)}{d(x)}.$$

Since x was an arbitrary point in $M^\epsilon - M$, and $\nabla d(x) = \frac{x - \pi(x)}{d(x)}$ is a continuous function of x over $M^\epsilon - M$, we conclude that d is continuously differentiable over $M^\epsilon - M$, completing the proof. **Q.E.D.**

We note the following corollary to Proposition 4.5.

Corollary 4.2 There exists $\epsilon' > 0$ such that $\text{cl}(M^{\epsilon'})$ is a smooth manifold with boundary $\text{bd}(M^{\epsilon'}) = \{x \in \mathbb{R}^n \mid d(x) = \epsilon'\}$.

Proof. Let ϵ' such that $0 < \epsilon' < \epsilon$. From Proposition 4.5, $d : M^\epsilon - M \mapsto (0, \epsilon)$ is

a continuously differentiable function with $\nabla d(y) = \frac{y - \pi(y)}{d(y)}$ for every $y \in M^\epsilon - M$. Since $\nabla d(y) \neq 0$ for every y such that $d(y) = \epsilon'$, by Lemma 4.1,

$$\text{cl}(M^{\epsilon'}) = \{x \in \mathbb{R}^n \mid d(x) \leq 0\}$$

is a smooth manifold with boundary. Furthermore, the boundary is characterized by

$$\text{bd}(M^{\epsilon'}) = \{x \in \mathbb{R}^n \mid d(x) = \epsilon'\},$$

completing the proof. **Q.E.D.**

For the rest of the subsection, we assume that $\epsilon > 0$ is a sufficiently small fixed scalar such that it also satisfies the result of Corollary 4.2.

Given $y \in M^\epsilon$, we define $\lambda(y) \in \mathbb{R}^{|I(\pi(y))|}$ to be the unique vector that satisfies

$$y - \pi(y) = G(\pi(y))\lambda(y). \quad (4.25)$$

We also define $H(y)$ as

$$H(y) = \sum_{i \in I(\pi(y))} \lambda_i(y) H_{g_i}(\pi(y)) \quad (4.26)$$

and we adopt the notation

$$[X]_Y = Y^T X Y$$

where X and Y are matrices with appropriate dimensions. If Y is invertible, we also adopt

$$[X]_Y = Y^{-1} X Y.$$

Proposition 4.6 Let y be a vector in $\text{ri}(N^\epsilon(\pi(y)))$. Then, π is differentiable at y .

Moreover, we have ¹³

$$[\nabla\pi(y)]|_{C(\pi(y))} = \begin{bmatrix} 0 & 0 \\ 0 & (I + [H(y)]\|_{V(\pi(y))})^{-1} \end{bmatrix},$$

where $C(x) = [G(x), V(x)]$ is a normal-tangent basis for $x \in M$ (see Section 4.3.1).

Proof. We assume without loss of generality that $I(\pi(y)) = \{1, 2, \dots, I_y\}$ where $I_y = |I(\pi(y))|$. We will prove the proposition using the implicit function theorem. Let $f : M^\epsilon \times \mathbb{R}^n \times \mathbb{R}^{I_y} \mapsto \mathbb{R}^{n+I_y}$ such that

$$\begin{aligned} f_{\{1,2,\dots,n\}}(v, p, \gamma) &= v - p - \sum_{i \in I(\pi(y))} \gamma_i \nabla g_i(p), \\ f_{n+j}(v, p, \gamma) &= g_j(p) \text{ for } j \in I(\pi(y)). \end{aligned}$$

Then f is a differentiable function since the g_i are twice continuously differentiable. For $a = (y, \pi(y), \lambda(y))$, we have $f(a) = 0$. Denote by $J(y)$ the Jacobian $\nabla_{p,\gamma} f$ evaluated at a . Then

$$J(y) = \begin{bmatrix} J_{UL}(y) & G(\pi(y)) \\ G(\pi(y))^T & 0 \end{bmatrix} \quad (4.27)$$

where $J_{UL}(y) = -I - H(y)$.

We first claim that $J_{UL}(y)$ is negative definite. From the proof of Lemma 4.3, we know that ϵ was chosen sufficiently small that

$$\sum_{k \in I(\pi(y))} |\lambda_k(y)| \leq \frac{1}{2n(H_m + 2\mu)}. \quad (4.28)$$

where

$$H_m = \max_{i \in I} \max_{x \in M} \|H_{g_i}(x)\|,$$

and $\mu > 0$ is a constant scalar. For a matrix A , let A^{ij} denote its entry at i^{th} row

¹³This result agrees with similar formulas obtained by Holmes [28] with the gauge function when M is assumed to be convex.

and j^{th} column. Then from the definition of H_m , we have

$$|H_{g_k}(\pi(y))^{ij}| \leq H_m, \quad \forall i, j \in \{1, 2, \dots, n\}.$$

Then for $i \in \{1, 2, \dots, n\}$, we have

$$\begin{aligned} (J_{UL}(y))^{ii} - \sum_{j \in \{1, 2, \dots, n\} - \{i\}} |J_{UL}(y)^{ij}| &= -1 + \sum_{k \in I(\pi(y))} \lambda_k(y) H_{g_k}(\pi(y))^{ii} - \\ &\quad \sum_{j \in \{1, 2, \dots, n\} - \{i\}} \left| \sum_{k \in I(\pi(y))} \lambda_k(y) H_{g_k}(\pi(y))^{ij} \right| \\ &\leq -1 + \sum_{j \in \{1, 2, \dots, n\}} \sum_{k \in I(\pi(y))} |\lambda_k(y)| H_m \\ &\leq -1 + \sum_{j \in \{1, 2, \dots, n\}} \frac{1}{2n(H_m + 2\mu)} H_m \\ &< -1 + \sum_{j \in \{1, 2, \dots, n\}} 1/n \\ &< -1 + 1 = 0 \end{aligned}$$

where we used Eq. (4.28) to get the second inequality. Thus, $J_{UL}(y)$ is strictly diagonally negative dominant and hence is negative definite. We next claim that $J(y)$ is nonsingular. Assume the contrary, that there exists $(p, \gamma) \neq 0$ such that $J(y)(p, \gamma)^T = 0$. Then,

$$J_{UL}(y)p^T + G(\pi(y))\gamma^T = 0 \tag{4.29}$$

$$G(\pi(y))^T p^T = 0. \tag{4.30}$$

Pre-multiplying Eq. (4.29) by p and using Eq. (4.30), we obtain;

$$pJ_{UL}(y)p^T = 0.$$

Since $J_{UL}(y)$ is strictly negative definite, it follows that $p = 0$. Then, from Eq. (4.29),

$G(\pi(y))\gamma = 0$ and since the columns of $G(\pi(y))$ are linearly independent, we obtain $\gamma = 0$. Thus, $(p, \gamma) = 0$, which is a contradiction. Hence, $J(y)$ is nonsingular.

Then, the implicit function theorem applies to $f(v, p, \gamma)$ and there exists open sets $D^1 \subset M^\epsilon$, $D^2 \subset \mathbb{R}^n$, $D^3 \subset \mathbb{R}^{I(y)}$ such that

$$a = (y, \pi(y), \lambda(y)) \in (D^1 \times D^2 \times D^3)$$

and there exists unique differentiable functions $p : D^1 \mapsto D^2$, $\gamma : D^1 \mapsto D^3$ such that

$$f(v, p(v), \gamma(v)) = 0$$

for all $v \in D^1$.

We next show that there exists an open set $D \subset \mathbb{R}^n$ containing y such that $p(v) = \pi(v)$ and $\gamma(v) = \lambda(v)$ for all $v \in D$. Note that $p(y) = \pi(y)$ and $\gamma(y) = \lambda(y)$. Since $y \in \text{ri}(N^\epsilon(\pi(y)))$, we have $\lambda(y) > 0$. Then $\gamma(y) = \lambda(y) > 0$, and the continuity of γ implies that there exists an open set $D^\gamma \subset D^1$ containing y such that $\gamma(v) > 0$ for $v \in D^\gamma$. Similarly, $g_j(p(y)) = g_j(\pi(y)) < 0$ for all $j \notin I(\pi(y))$, thus there exists an open set $D^g \subset D^1$ containing y such that $g_j(p(v)) < 0$ for all $j \notin I(\pi(y))$ and $v \in D^g$. Let

$$D = D^\gamma \cap D^g.$$

Then $y \in D$. For $v \in D$ we have

$$\begin{aligned} v &= p(v) + \sum_{i \in I(\pi(y))} \gamma_i(v) \nabla g_i(p(v)) \\ g_i(p(v)) &= 0, \quad \forall i \in I(y) \\ g_j(p(v)) &< 0, \quad \forall j \notin I(y). \end{aligned} \tag{4.31}$$

Then we have $p(v) \in M$ and $I(p(v)) = I(\pi(y))$ and it follows from Eq. (4.31) that $v \in N^\epsilon(p(v))$. Therefore, from Lemma 4.4 and the definition in (4.25), we have $p(v) = \pi(v)$ and $\gamma(v) = \lambda(v)$ for all $v \in D$.

Since p is a differentiable function over a neighborhood D of y , we conclude that

π is differentiable at y as desired. Moreover, by the implicit function theorem, we have the following expression for the Jacobian of $(\pi; \gamma)$ at y ;

$$\begin{aligned}\nabla(p, \gamma)(y) &= -(\nabla_{(p, \gamma)} f(a))^{-1} \nabla_v f(a) \\ &= -J(y)^{-1} I^{(n+I_y, n)}\end{aligned}$$

where for positive integers n, m

$$I^{(n, m)} = \begin{cases} \text{the } n \times n \text{ identity matrix, if } n = m, \\ \text{the } n \times m \text{ matrix } [I^{(n, n)} \ 0], \text{ if } n < m, \\ \text{the } n \times m \text{ matrix } [I^{(m, m)}, \ 0]^T, \text{ if } n > m. \end{cases}$$

For $y \in \text{ri}(N^\epsilon(\pi(y)))$, since $p = \pi$ in a neighborhood of y we have

$$\nabla \pi(y) = -I^{(n, n+I_y)} J(y)^{-1} I^{(n+I_y, n)}. \quad (4.32)$$

For notational simplicity, we fix y and denote $V = V(\pi(y))$, $G = G(\pi(y))$, $C = C(\pi(y))$, $H = H(y)$, $J = J(y)$, and $J_{UL} = J_{UL}(y)$. By Eq. (4.32), we have

$$[\nabla \pi(y)]|_C = -C^{-1} I^{(n, n+I_y)} J^{-1} I^{(n+I_y, n)} C. \quad (4.33)$$

First note that

$$I^{(n+I_y, n)} C = \begin{bmatrix} \nabla g_i(\pi(y)) & \dots & v_j & \dots \\ 0 & \dots & 0 & \dots \end{bmatrix}.$$

Let e^k denote the unit vector in \mathbb{R}^{n+I_y} . Then

$$J e^{n+i} = \begin{bmatrix} \nabla g_i(\pi(y)) \\ 0 \end{bmatrix},$$

thus

$$J^{-1} \begin{bmatrix} \nabla g_i(\pi(y)) \\ 0 \end{bmatrix} = e^{n+i}. \quad (4.34)$$

Let

$$J^{-1} \begin{bmatrix} v_j \\ 0 \end{bmatrix} = \begin{bmatrix} x_j \\ y_j \end{bmatrix}. \quad (4.35)$$

Then,

$$J \begin{bmatrix} x_j \\ y_j \end{bmatrix} = \begin{bmatrix} v_j \\ 0 \end{bmatrix}$$

and thus

$$J_{UL}x_j + Gy_j = v_j, \quad (4.36)$$

and

$$G^T x_j = 0. \quad (4.37)$$

Since columns of V span the space of vectors orthogonal to each column of G , Eq. (4.37) implies that $x_j = V\beta_j$ for some $\beta_j \in \mathbb{R}^{n-I_y}$. Then, pre-multiplying Eq. (4.36) by V^T , we have

$$V^T J_{UL} V \beta_j + V^T G y_j = V^T v_j$$

and using the fact that $V^T G = 0$ and that $V^T J_{UL} V$ is invertible¹⁴, we get

$$\beta_j = (V^T J_{UL} V)^{-1} e_j \quad (4.38)$$

hence β_j is the j^{th} column of $(V^T J_{UL} V)^{-1}$. Let $\beta = [\beta_j]_{j \in \{1, 2, \dots, n-I_y\}}$ and $Y = [y_j]_{j \in \{1, 2, \dots, n-I_y\}}$. Then, using Equations (4.34), (4.35), and (4.38)

$$J^{-1} I^{(n+I_y, n)} C = \begin{bmatrix} 0 & V\beta \\ I^{(I_y, I_y)} & Y \end{bmatrix} = \begin{bmatrix} 0 & V (V^T J_{UL} V)^{-1} \\ I^{(I_y, I_y)} & Y \end{bmatrix}.$$

¹⁴Since J_{UL} is strictly negative definite, $V^T J_{UL} V$ is strictly negative definite, and hence is invertible.

Substituting this expression in Eq. (4.33) yields

$$\begin{aligned}
[\nabla\pi(y)]|_C &= -C^{-1}I^{(n,n+I_y)} \begin{bmatrix} 0 & V(V^T J_{UL} V)^{-1} \\ I^{(I_y, I_y)} & Y \end{bmatrix} \\
&= -C^{-1} \begin{bmatrix} 0 & V(V^T J_{UL} V)^{-1} \end{bmatrix} \\
&= C^{-1}[G \ V] \begin{bmatrix} 0 & 0 \\ 0 & (V^T(-J_{UL})V)^{-1} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & (V^T(I+H)V)^{-1} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & (I+V^T H V)^{-1} \end{bmatrix},
\end{aligned}$$

showing the desired relation. **Q.E.D.**

4.3.2.2 Extension Theorem

Theorem 4.2 Let M be a region given by (4.13). Let U be an open set containing M and $F : U \mapsto \mathbb{R}^n$ be a continuously differentiable function. Let $F_m = \max_{x \in M} F(x)$ and let $K \in \mathbb{R}$. Let $F_K : \text{cl}(M^\epsilon) \mapsto \mathbb{R}^n$ be defined as

$$F_K(y) = F(\pi(y)) + K(y - \pi(y)).$$

Then, we have the following.

- (i) For any $K \in \mathbb{R}$, F_K is a continuous function.
 - (ii) For any $K \in \mathbb{R}$, F_K is differentiable for all $y \in M^\epsilon$ such that $y \in \text{ri}(N^\epsilon(\pi(y)))$.
- Moreover, the Jacobian in the tangent-normal coordinates of $\pi(y)$ is

$$[\nabla F_K(y)]|_{C(\pi(y))} = \begin{bmatrix} KI & S \\ 0 & [\nabla F(\pi(y)) + KH(y)] \|_{V(\pi(y))} (I + [H(y)] \|_{V(\pi(y))})^{-1} \end{bmatrix}.$$

for some $I_y \times (n - I_y)$ matrix S . Furthermore,

$$\det(\nabla F_K(y)) = K^{I_y} \det([\nabla F(\pi(y)) + KH(y)] \|_{V(\pi(y))}) \\ \det(I + [H(y)] \|_{V(\pi(y))})^{-1}.$$

Also, if $K > 0$, then

$$\text{sign}(\det(\nabla F_K(y))) = \text{sign}(\det([\nabla F(\pi(y)) + KH(y)] \|_{V(\pi(y))})).$$

(iii) If $K > K_m = \frac{F_m}{\epsilon}$, F_K points outward on the boundary of M^ϵ . In other words given $x \in \text{bd}(M)$, there exists a sequence $\epsilon_i \downarrow 0$ such that $x + \epsilon_i F(x) \notin M$ for all $i \in \mathbb{Z}^+$.

Proof. (i) Follows immediately since the projection function π is continuous.

(ii) Let $y \in \text{ri}(N^\epsilon(\pi(y)))$. Then F_K is differentiable at y since π is differentiable at y [cf. Proposition 4.6]. For notational simplicity, we fix y and denote $V = V(\pi(y))$, $G = G(\pi(y))$, $C = C(\pi(y))$, $H = H(y)$. Since $V^T G = 0$ and $V^T V = I$, we note that

$$[G \ V]^{-1} = \begin{bmatrix} R \\ V^T \end{bmatrix} \quad (4.39)$$

for some matrix R . We can write the Jacobian of F_K in tangent-normal coordinates

as

$$\begin{aligned}
[\nabla F_K(y)]|_C &= [\nabla F(\pi(y))]|_C [\nabla \pi(y)]|_C + K(I - [\nabla \pi(y)]|_C) \\
&= ([\nabla F(\pi(y))]|_C - KI) [\nabla \pi(y)]|_C + KI \\
&= [G \ V]^{-1} (\nabla F(\pi(y)) - KI) [G \ V] \begin{bmatrix} 0 & 0 \\ 0 & (I + V^T H V)^{-1} \end{bmatrix} + KI \\
&= \begin{bmatrix} .. & .. \\ .. & V^T (\nabla F(\pi(y)) - KI) V \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & (I + V^T H V)^{-1} \end{bmatrix} \\
&\quad + \begin{bmatrix} KI & 0 \\ 0 & K(I + V^T H V)(I + V^T H V)^{-1} \end{bmatrix}
\end{aligned}$$

where, to get the last equation, we used Eq. (4.39) and the fact that $(V^T(I + H)V)$ is invertible. We then have

$$\begin{aligned}
[\nabla F_K(y)]|_C &= \begin{bmatrix} 0 & S \\ 0 & V^T (\nabla F(\pi(y)) - KI) V (I + V^T H V)^{-1} \end{bmatrix} \\
&\quad + \begin{bmatrix} KI & 0 \\ 0 & V^T (KI + KH) V (I + V^T H V)^{-1} \end{bmatrix} \\
&= \begin{bmatrix} KI & S \\ 0 & V^T (\nabla F(\pi(y)) - KI + KI + KH) V (I + V^T H V)^{-1} \end{bmatrix} \\
&= \begin{bmatrix} KI & S \\ 0 & V^T (\nabla F(\pi(y)) + KH) V (I + V^T H V)^{-1} \end{bmatrix}
\end{aligned}$$

as desired. The determinant then can be calculated as

$$\begin{aligned}
\det(\nabla F_K(y)) &= \det(KI^{I_y \times I_y}) \det(V^T (\nabla F(\pi(y)) + KH) V (I + V^T H V)^{-1}) \\
&= K^{I_y} \det(V^T (\nabla F(\pi(y)) + KH) V) \det(I + V^T H V)^{-1}
\end{aligned}$$

as desired. We have noted that $V^T(-I - H)V$ is negative definite. Then, $I + V^T H V$

is positive definite and thus $(I + V^T H V)^{-1}$ is positive definite. Then

$$\det (I + V^T H V)^{-1} > 0$$

and for $K > 0$ we have

$$\text{sign} (\det (\nabla F_K(y))) = \text{sign} (\det (V^T (\nabla F(\pi(y)) + K H) V))$$

as desired.

(iii) We note from Corollary 4.2 that

$$\text{bd}(M^\epsilon) = \{y \in \mathbb{R}^n \mid d(x) = \epsilon\}.$$

Let $y \in \text{bd}(M^\epsilon)$. From Proposition 4.5, we have

$$\nabla d(y) = \frac{y - \pi(y)}{d(y)} \neq 0$$

thus

$$\text{cl}(M^\epsilon) = \{x \in \mathbb{R}^n \mid d(x) \leq \epsilon\}$$

satisfies the LICQ condition [cf. Section 4.3.1]. Then v points outward at $y \in \text{bd}(M^\epsilon)$ if and only if

$$v^T \nabla d(y) = v^T \frac{y - \pi(y)}{d(y)} > 0.$$

We have

$$\begin{aligned} F_K(y)^T (y - \pi(y)) &= F(\pi(y))^T (y - \pi(y)) + K (y - \pi(y))^T (y - \pi(y)) \\ &\geq -F_m \epsilon + K \epsilon^2 \\ &> \epsilon \left(K - \frac{F_m}{\epsilon} \right). \end{aligned}$$

If $K > \frac{F_m}{\epsilon}$, then $F_K(y)^T (y - \pi(y)) > 0$ for all $y \in \text{bd}(M^\epsilon)$ and hence F_K points outward on the boundary of M^ϵ as desired. This completes the proof of the theorem.

Q.E.D.

The following proposition establishes that there exists a one-to-one correspondence between the zeros of F_K over M^ϵ and the critical points of F over M .

Proposition 4.7 Let $K > K_m = \frac{F_m}{\epsilon}$ and $F_K : M^\epsilon \mapsto \mathbb{R}^n$ be the extension defined in Theorem 4.2. Let

$$Z = \{z \in M^\epsilon \mid F_K(z) = 0\}.$$

Let $s : \text{Cr}(F, M) \mapsto \mathbb{R}^n$ such that

$$s(x) = x - \frac{F(x)}{K} = x + G(x) \frac{\theta(x)}{K}.$$

Then, s is a one-to-one and onto function from $\text{Cr}(F, M)$ to Z , with inverse equal to $\pi|_Z$.

Proof. Clearly, $s(x) - x \in N_M(x)$. Also, since

$$\|s(x) - x\| = \left\| \frac{1}{K} G(x) \theta(x) \right\| = \frac{\|F(x)\|}{K} < \frac{\|F(x)\|}{F_m} \leq \epsilon,$$

we have

$$s(x) \in N^\epsilon(x). \tag{4.40}$$

Then it follows from Proposition 4.4 that $\pi(s(x)) = x$. Also, it follows from the definition of λ that $\lambda(s(x)) = \frac{\theta(x)}{K}$. We first show that $s(x) \in Z$ for all $x \in \text{Cr}(F, M)$.

Note that

$$\begin{aligned} F_K(s(x)) &= F(\pi(s(x))) + K(s(x) - x) \\ &= F(x) + K \left(\frac{1}{K} G(x) \theta(x) \right) \\ &= F(x) + G(x) \theta(x) = 0. \end{aligned}$$

Thus $s(x) \in Z$. We next show that $s : \text{Cr}(F, M) \mapsto Z$ is onto. To see this, let $y \in Z$.

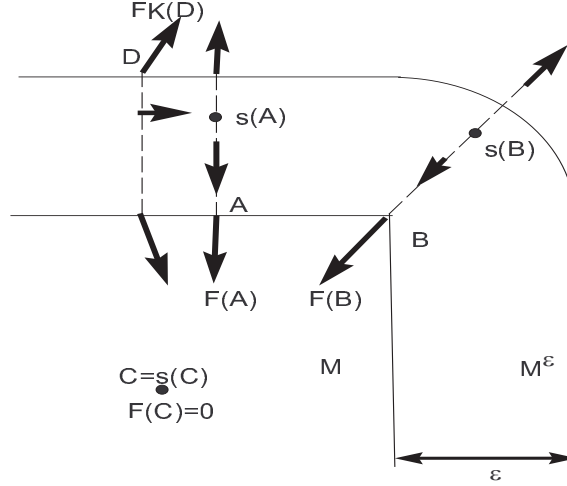


Figure 4.8: A,B,C are the generalized critical points of F , whereas $s(A)$, $s(B)$, and $s(C)$ are the critical points in the usual sense of the extended function F_K . In other words, s converts the generalized critical points of F to critical points of F_K .

Then,

$$F_K(y) = F(\pi(y)) + K(y - \pi(y)) = 0$$

implies

$$-F(\pi(y)) = K(y - \pi(y)) = KG(\pi(y))\lambda(y).$$

Since $K\lambda(y) \geq 0$, we have $-F(\pi(y)) \in N_M(\pi(y))$ and thus $\pi(y) \in \text{Cr}(F, M)$. Moreover,

$$s(\pi(y)) = \pi(y) - \frac{F(\pi(y))}{K} = \pi(y) + \frac{K(y - \pi(y))}{K} = y,$$

and thus $s : \text{Cr}(F, M) \mapsto Z$ is onto.

We finally show that s is one-to-one. Assume that there exists $x, x' \in \text{Cr}(F, M)$ such that $y = s(x) = s(x')$. Then $y \in N^\epsilon(x)$ and $y \in N^\epsilon(x')$ [cf. Eq. (4.40)]. From Lemma 4.1, $x = x'$ and thus s is one-to-one. We conclude that s is a one-to-one and onto function from $\text{Cr}(F, M)$ to Z . **Q.E.D.**

The following Lemma relates the index of a generalized critical point to the Poincare-Hopf index of the zero of the extended function.

Lemma 4.7 Let $K > K_m = F_m/\epsilon$ and $F_K : M^\epsilon \mapsto \mathbb{R}^n$ be the extension defined in Theorem 4.2. If $x \in \text{Cr}(F, M)$ is complementary and non-degenerate, then F_K is differentiable at $s(x)$. Moreover,

$$\text{ind}_F(x) = \text{sign}(\det(\nabla F_K(s(x)))) . \quad (4.41)$$

Proof. If $x \in \text{Cr}(F, M)$ is complementary, then $\theta(x) > 0$. Thus $\lambda(s(x)) = \frac{\theta(x)}{K} > 0$. Therefore, $s(x) \in \text{ri}(N^\epsilon(x))$, which by Proposition 4.2 implies that F_K is differentiable at $s(x)$. Furthermore, we have

$$\begin{aligned} \text{sign}(\det(\nabla F_K(s(x)))) &= \text{sign}(\det(V(x)^T(\nabla F(x) + KH(s(x)))V(x))) \\ &= \text{sign}\left(\det\left(V(x)^T\left(\nabla F(x) + \sum_{i \in I(x)} K\lambda_i(s(x))H_{g_i}(x)\right)V(x)\right)\right) \\ &= \text{sign}\left(\det\left(V(x)^T\left(\nabla F(x) + \sum_{i \in I(x)} \theta_i(x)H_{g_i}(x)\right)V(x)\right)\right) \\ &= \text{sign}(\det(\Gamma(x))) \\ &= \text{ind}_F(x) \end{aligned}$$

as desired. **Q.E.D.**

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let $K > K_m = F_m/\epsilon$ and $F_K : \text{cl}(M^\epsilon) \mapsto \mathbb{R}^n$ be the extension defined in Theorem 4.2. Let

$$Z = \{z \in \mathbb{R}^n \mid F_K(z) = 0\}.$$

By Corollary 4.2, $\text{cl}(M^\epsilon)$ is a smooth manifold with boundary. Moreover, by Proposition 4.2, F_K points outward on the boundary of M^ϵ and is differentiable at every

$y \in Z$ since $\pi(y) \in \text{Cr}(F, M)$ is complementary. Then, the Poincare-Hopf Theorem II applies to F_K and we have

$$\sum_{z \in Z} \text{sign}(\det(\nabla F_K(z))) = \chi(\text{cl}(M^\epsilon)).$$

Then using Proposition 4.7

$$\begin{aligned} \sum_{x \in \text{Cr}(F, M)} \text{ind}_F(x) &= \sum_{x \in \text{Cr}(F, M)} \text{sign}(\det(\Gamma(x))) \\ &= \sum_{s(x) \in Z} \text{sign}(\det(\nabla F_K(s(x)))) \\ &= \chi(\text{cl}(M^\epsilon)). \end{aligned} \tag{4.42}$$

To complete the proof, we need the following lemma.

Lemma 4.8 $\text{cl}(M^\epsilon)$ is homotopy equivalent to M , i.e. there exists continuous functions $f : \text{cl}(M^\epsilon) \mapsto M$ and $g : M \mapsto \text{cl}(M^\epsilon)$ such that $f \circ g$ is homotopic to i_M , and $g \circ f$ is homotopic to $i_{\text{cl}(M^\epsilon)}$ [cf. Definition 4.1], where $i_X : X \mapsto X$ denotes the identity function on some set X . In particular, $\chi(\text{cl}(M^\epsilon)) = \chi(M)$.

Proof. Let $i_X : X \mapsto X$ denote the identity map on some set X . Let $f = \pi|_{\text{cl}(M^\epsilon)} : \text{cl}(M^\epsilon) \mapsto M$ and $g = i_M : M \mapsto M$. Then, f and g are continuous and

$$f \circ g(x) = f(g(x)) = \pi(x) = x, \quad \text{for all } x \in M.$$

Then, $f \circ g = i_M$ and thus is homotopic to i_M . We have

$$g \circ f(x) = g(f(x)) = \pi(x), \quad \text{for all } x \in \text{cl}(M). \tag{4.43}$$

Let $F(\text{cl}(M^\epsilon) \times [0, 1]) \mapsto \mathbb{R}^n$ such that

$$F(x, \xi) = (1 - \xi)\pi(x) + \xi x.$$

F is continuous since π is continuous over $\text{cl}(M^\epsilon)$. We have,

$$F(x, 0) = \pi(x) = g \circ f(x), \quad \text{for all } x \in \text{cl}(M)$$

where we used Eq. (4.43), and

$$F(x, 1) = x = i_{\text{cl}(M)}(x), \quad \text{for all } x \in \text{cl}(M).$$

Thus, F is a homotopy between $g \circ f$ and $i_{\text{cl}(M)}$, which implies that $g \circ f$ is homotopic to $i_{\text{cl}(M)}$. Then f and g satisfy the conditions for homotopy equivalence of $\text{cl}(M^\epsilon)$ and M , which implies that $\text{cl}(M^\epsilon)$ is homotopy equivalent to M as desired ¹⁵.

Since the Euler characteristic is invariant for sets that are homotopy equivalent to each other (see, for example, Massey [37]), we have $\chi(\text{cl}(M^\epsilon)) = \chi(M)$, completing the proof of Lemma 4.8 **Q.E.D.**

Combining the result of Lemma 4.8 with Eq. (4.42), we complete the proof of Theorem 4.1. **Q.E.D.**

4.4 Generalized Poincare-Hopf Theorem and the Uniqueness Problem

We first present an important application of Theorem 4.1 to nonconvex optimization theory. The following definition generalizes the notions of stationary point and non-degeneracy.

Definition 4.6 Consider a compact region M defined by (4.13). Let U be an open set containing M and $f : U \mapsto \mathbb{R}$ be a twice continuously differentiable function.

¹⁵In fact, we proved the stronger result that M is a *strong deformation retract* of M^ϵ , i.e. there exists a homotopy $F : M^\epsilon \times [0, 1] \mapsto M^\epsilon$ such that $F(x, 0) = x$ and $F(x, 1) \in M$ for all $x \in M^\epsilon$ (cf. Kosniowski [32]).

(a) We say that $x \in M$ is a *KKT point of f over M* ¹⁶ if there exists $\mu_i \geq 0$ such that

$$\nabla f(x) + \sum_{i \in I(x)} \mu_i \nabla g_i(x) = 0. \quad (4.44)$$

We denote the set of KKT points of f over M with $\text{KKT}(f, M)$.

(b) For $x \in \text{KKT}(f, M)$, we define $\mu(x) \geq 0$ to be the unique vector in $\mathbb{R}^{I(x)}$ that satisfies Eq. (4.44). We say that x is a *complementary KKT point* if $\mu(x) > 0$.

(c) We define

$$\text{KKT}_f(x) = V(x)^T \left(H_f(x) + \sum_{i \in I(x)} \mu_i(x) H_{g_i}(x) \right) V(x)$$

where $V(x)$ denotes the change of coordinates matrix from standard coordinates to tangent coordinates as given by Section 4.3.1. We say that x is a *non-degenerate KKT point* if $\text{KKT}_f(x)$ is a non-singular matrix.

(d) Let x be a complementary and non-degenerate KKT point of f over M . We define the *KKT-index* of f at x as

$$\text{KKTind}_f(x) = \text{sign}(\det(\text{KKT}_f(x))).$$

(e) We say that f is a *KKT-non-degenerate function over M* if every KKT point of f over M is non-degenerate.

The following generalization of Proposition 4.1 establishes the Karush-Kuhn-Tucker necessary and sufficient optimality conditions in terms of the constructs defined above. It establishes a connection between KKT points of f and local minima of f , hence justifies the above definition and motivates the terminology used.

Proposition 4.8 Consider a region M given by (4.13). Let U be an open set containing M and $f : U \mapsto \mathbb{R}$ a twice continuously differentiable function.

(i) If $x \in M$ is a local minimum of f , then x is a KKT point of f and $\text{KKT}_f(x)$ is a

¹⁶*KKT point* is short for *minimum type Karush-Kuhn-Tucker stationary point*.

positive semi-definite matrix.

(ii) If $x \in M$ is a complementary KKT point and $\text{KKT}_f(x)$ is a positive definite matrix, then x is a strict local minimum.

Proof. (i) Since x is a local minimum of f over the regular set M , we have, by the necessary optimality conditions (cf. Proposition 3.3.1 of Bertsekas [11]) that there exists a unique Lagrange multiplier vector $\mu \in \mathbb{R}^{I(x)}$ such that Eq. (4.44) is satisfied. Then x is a KKT point of f . By the necessary optimality conditions, we also have

$$y^T(H_f(x) + \sum_{i \in I(x)} \mu_i(x)H_{g_i}(x))y \geq 0, \quad (4.45)$$

for all $y \in \mathbb{R}^n$ such that

$$G(x)^T y = 0.$$

Given $z \in \mathbb{R}^{n-|I(x)|}$, let $y = V(x)z$. Then,

$$G(x)^T y = G(x)^T V(x)z = 0,$$

since $G(x)$ and $V(x)$, by definition, have orthogonal columns [cf. Section 4.3.1]. Then, by Eq. (4.45), we have

$$\begin{aligned} 0 &\leq y^T(H_f(x) + \sum_{i \in I(x)} \mu_i(x)H_{g_i}(x))y \\ &= (V(x)z)^T(H_f(x) + \sum_{i \in I(x)} \mu_i(x)H_{g_i}(x))V(x)z \\ &= z^T V(x)^T(H_f(x) + \sum_{i \in I(x)} \mu_i(x)H_{g_i}(x))V(x)z \\ &= z^T \text{KKT}_f(x)z \end{aligned}$$

which implies that $\text{KKT}_f(x)$ is positive semi-definite as desired.

(ii) We have that $x \in M$ satisfies Eq. (4.44) with $\mu > 0$. Moreover, given $y \in \mathbb{R}^n$ such that $y \neq 0$ and $G(x)^T y = 0$, we have, by definition of $V(x)$ and $G(x)$ [cf. Section 4.3.1], that there exists $v \in \mathbb{R}^{n-|I(x)|}$ such that $y = V(x)v$. $v \neq 0$ since $y \neq 0$, then

since $\text{KKT}_f(x)$ is strictly positive definite, we have

$$\begin{aligned}
0 &< v^T \text{KKT}_f(x) v \\
&= v^T V(x)^T (H_f(x) + \sum_{i \in I(x)} \mu_i(x) H_{g_i}(x)) V(x) v \\
&= (V(x) v)^T (H_f(x) + \sum_{i \in I(x)} \mu_i(x) H_{g_i}(x)) V(x) v \\
&= y^T (H_f(x) + \sum_{i \in I(x)} \mu_i(x) H_{g_i}(x)) y
\end{aligned}$$

Then, from the second order sufficiency optimality conditions (cf. Proposition 3.3.2 Bertsekas [11]), x is a strict local minimum, as desired. **Q.E.D.**

We have the following corollary to the proposition.

Corollary 4.3 Consider a region M given by (4.13). Let U be an open set containing M and $f : U \mapsto \mathbb{R}$ be a twice continuously differentiable KKT-non-degenerate function over M . If x is a local minimum of f over M , then $\text{KKTind}_f(x) = 1$.

Proof. By Proposition 4.8, $\text{KKT}_f(x)$ is positive semi-definite. Since f is KKT-non-degenerate, we also have that $\text{KKT}_f(x)$ is positive definite and thus

$$\det(\text{KKT}_f(x)) > 0,$$

which implies that $\text{KKTind}_f(x) = 1$. **Q.E.D.**

As suggested by the similarity between Definition 4.2 and Definition 4.6, Karush-Kuhn-Tucker points of f over M correspond to the generalized critical points of ∇f over M . The following proposition regarding KKT points of f over M follows from Theorem 4.1 exactly the same way Proposition 4.2 follows from the Poincare-Hopf Theorem. It establishes that the pattern presented in Proposition 4.2 regarding the stationary points holds for the more general KKT stationary points.

Proposition 4.9 Consider a non-empty compact region M given by (4.13). Let U be an open set containing M and $f : U \mapsto \mathbb{R}$ be a twice continuously differentiable

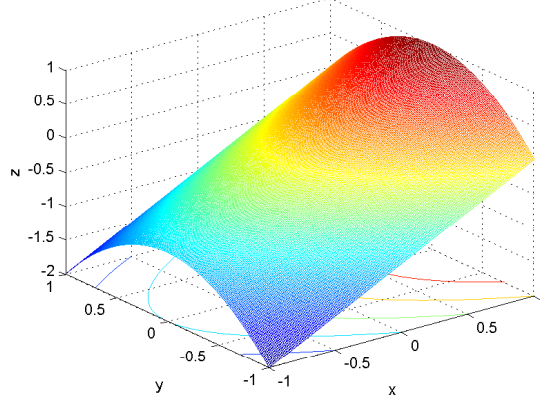


Figure 4.9: Illustration of Proposition 4.9. $(-1, -1)$ and $(-1, 1)$ are KKT points which are also strict minimum, hence they have KKT-index equal to 1. The only other *KKT* point is the boundary saddle $(-1, 0)$, with KKT-index equal to -1 . The sum of KKT-indices add up to the Euler characteristic of the region.

function. Assume that f is KKT-non-degenerate over M . Then, we have

$$\chi(M) = \sum_{x \in \text{KKT}(f, M)} \text{KKTind}_f(x). \quad (4.46)$$

Before we give the proof, we illustrate this result with the following example.

Example 4.3 Let

$$M = \{(x, y) \in \mathbb{R}^2 \mid x \in [-1, 1], y \in [-1, 1]\}.$$

Let $f : M \mapsto \mathbb{R}$ given by

$$f(x, y) = x - y^2.$$

Then, M is a region given by (3.10), f has three KKT-points at $a = (-1, -1)$, $b = (-1, 1)$, and $c = (-1, 0)$, all of which are non-degenerate and complementary, thus the above proposition applies. We note that a and b are strict minima and, by Proposition 4.8, have KKT-indices equal to 1. On the other hand, c is a KKT-point that is not a strict minimum. At c , the function f is decreasing in boundary directions that are tangent to the region, which implies that c has KKT-index equal to -1 (which can

be checked algebraically)¹⁷. Then, the sum of the KKT indices of the critical points is $1 + 1 - 1 = 1$ which is the Euler characteristic of the region M .

In [29], Jongen et al. prove Proposition 4.9 by generalizing Morse Theory to constrained regions. In the Morse Theory approach, one tracks the change in the topology of the lower level sets

$$L_c = \{x \in M \mid f(x) \leq c\}$$

as c changes from

$$a = \min_{x \in M} f(x)$$

to

$$b = \max_{x \in M} f(x).$$

$c \in \mathbb{R}$ is said to be a *regular value* if there is no $x \in \text{KKT}(f, M)$ such that $f(x) = c$, and a *critical value* otherwise. It can then be shown that the topology of the level sets do not change when one crosses a regular value, while the topology changes in a unique way up to an index¹⁸ when one crosses a critical value¹⁹. This allows one to relate the topology of

$$M = L_b$$

with the KKT-indices of KKT points of f over M , yielding results similar to Eq. (4.46).

Here, we present an alternative proof using Theorem 4.1.

¹⁷We call points like c *boundary saddle points*. More precisely, a boundary saddle point is a KKT point, hence a candidate for minima, but is not a minimum since f is decreasing along the boundary directions tangent to the region. In general, the index of a boundary saddle would be $(-1)^k$ where k is the dimension of the subspace over which f is decreasing at c . In this example, $k = 1$ and thus the index is -1 .

¹⁸This is called the Morse index and is defined as the number of negative eigenvalues of $\text{KKT}_f(x)$. Thus, if we denote the Morse index of a KKT point x by $M(x)$, we have $(-1)^{M(x)} = \text{KKTind}_f(x)$.

¹⁹To be more precise, let $c^- < c$ and $c^+ > c$ denote scalars sufficiently close to c . It can be shown that when c is a regular value, L_{c^-} and L_{c^+} are homeomorphic, whereas when c is a critical value (and when there is a unique $x \in \text{KKT}(f, M)$ with $f(x) = c$), L_{c^+} is topologically equivalent to L_{c^-} attached to the $M(x)$ -dimensional unit ball, where $M(x)$ denotes the Morse index of x . See [29] for details of this argument.

Proof of Proposition 4.9. First we note that $\nabla f : U \mapsto R^n$ is a continuously differentiable function on M and the KKT points of f over M are precisely the generalized critical points of ∇f over M with $\theta(x) = \mu(x)$. Thus we have

$$\text{KKT}(f, M) = \text{Cr}(\nabla f, M). \quad (4.47)$$

Let $x \in \text{Cr}(\nabla f, M)$. We have $\mu(x) > 0$ since x is a complementary KKT point of f . Then $\theta(x) = \mu(x) > 0$ and x is a complementary critical point of ∇f . Since x is a non-degenerate KKT point, $\text{KKT}_f(x)$ is non-singular. Then since

$$\begin{aligned} \text{KKT}_f(x) &= \det \left(V(x)^T \left(H_f(x) + \sum_{i \in I(x)} \mu_i(x) H_{g_i}(x) \right) V(x) \right) \\ &= \det \left(V(x)^T \left(\nabla(\nabla f(x)) + \sum_{i \in I(x)} \theta_i(x) H_{g_i}(x) \right) V(x) \right) \\ &= \Gamma(x), \end{aligned} \quad (4.48)$$

$\Gamma(x)$ is also non-singular and hence x is a non-degenerate critical point of ∇f . Then applying Theorem 4.1 with $F = \nabla f$, we have

$$\chi(M) = \sum_{x \in \text{Cr}(f, M)} \text{ind}_{\nabla f}(x). \quad (4.49)$$

Eq. (4.48) further implies

$$\text{KKTind}_f(x) = \text{sign}(\det(\text{KKT}_f(x))) = \text{sign}(\det(\Gamma(x))) = \text{ind}_{\nabla f}(x).$$

Combining this equation with Equations (4.47) and (4.49), we have

$$\chi(M) = \sum_{x \in \text{KKT}(f, M)} \text{KKTind}_f(x)$$

as desired. **Q.E.D.**

We present an important corollary of this Proposition regarding the uniqueness

problem.

Corollary 4.4 Consider a non-empty compact region M given by (4.13). Let U be an open set containing M and $f : U \mapsto \mathbb{R}$ be a twice continuously differentiable function. Assume that f is KKT-non-degenerate over M and every KKT point of f over M has KKT-index equal to 1, i.e.

$$\text{KKTind}_f(x) = 1$$

for all $x \in \text{KKT}(f, M)$. Then, we have the following:

- (i) The number of KKT points of f over M is equal to $\chi(M)$.
- (ii) When $\chi(M) = 1$, f has a unique local minimum over M which is also the global minimum.

Proof. (i) From Proposition 4.9, we have

$$\chi(M) = \sum_{x \in \text{KKT}(f, M)} \text{KKTind}_f(x).$$

Using the fact that $\text{KKTind}_f(x) = 1$ for all $x \in \text{KKT}(f, M)$, we have

$$\chi(M) = |\text{KKT}(f, M)|$$

where $|\text{KKT}(f, M)|$ denotes the number of elements in $\text{KKT}(f, M)$, as desired.

(ii) From part (i) and the fact that $\chi(M) = 1$, $\text{KKT}(f, M)$ has a unique element. Since M is compact and f is continuous, f has a global minimum over M . Let x be a global minimum of f over M . From Proposition 4.8, $x \in \text{KKT}(f, M)$. This implies that x is the only local minimum of f over M , completing the proof of the corollary. **Q.E.D.**

We can now generalize Proposition 4.3 and prove uniqueness results which are comparable to the uniqueness results of Chapter 3. The following proposition is the index theory counter-part to Proposition 3.3 stated at the end of Chapter 3,

which makes a similar claim under a different topological assumption and additional differentiability assumptions.

Proposition 4.10 Let M be a non-empty compact set given by (4.13), $U \subset \mathbb{R}^n$ be an open set containing M and $f : U \mapsto \mathbb{R}$ be a twice continuously differentiable function. Assume that f is KKT-non-degenerate and $\chi(M) = 1$. If (f, M) satisfies (Q1) and (B2), then (f, M) satisfies (P2).

Proof. Consider a Karush-Kuhn-Tucker point of f over M , i.e. consider $x \in \text{KKT}(f, M)$. By Definition 4.6, we have

$$-\nabla f(x) \in N_M(x).$$

Since (f, M) satisfies (B2), we have $x \notin \text{bd}(M)$. Then, $x \in \text{int}(M)$, which implies $N_M(x) = \{0\}$, which further implies $\nabla f(x) = 0$. Hence the KKT stationary point x is a stationary point in the usual sense. Note that, in this case, we have

$$\text{KKTind}_f(x) = \text{sign}(\det(H_f(x))). \quad (4.50)$$

Since f satisfies (Q1), x is a strict local minimum. Then from Proposition 4.1, $H_f(x)$ is positive semi-definite. Since f is non-degenerate, $H_f(x)$ is also non-singular, and hence is positive definite. Then we have

$$\det(H_f(x)) > 0, \quad \forall x \in \text{KKT}(f, M).$$

and by Eq. (4.50)

$$\text{KKTind}_f(x) = 1, \quad \forall x \in \text{KKT}(f, M).$$

Then, Corollary 4.4 applies and we have that f has a unique local minimum over M . f satisfies (P2) as desired. **Q.E.D.**

We now establish uniqueness results with local conditions that treat the boundary

and interior stationary points uniformly, and that are, for this aspect, stronger than the uniqueness results of the previous chapter. For this purpose, consider the following conditions. Let M be a non-empty compact set given by (4.13), $U \subset \mathbb{R}^n$ be an open set containing M and $f : U \mapsto \mathbb{R}$ be a twice continuously differentiable function. Let

(Q2) Every KKT point of f over M is complementary and satisfies

$$\det(\text{KKT}_f(x)) > 0.$$

(Q3) Every KKT point of f over M is a strict local minimum.

We then have the following generalizations of Proposition 3.2.

Proposition 4.11 Let M be a non-empty compact set given by (4.13), $U \subset \mathbb{R}^n$ be an open set containing M and $f : U \mapsto \mathbb{R}$ be a twice continuously differentiable function. If (f, M) satisfies (Q2), then (f, M) satisfies (P2).

Proof. Note that (Q2) implies, in particular, that f is a KKT-non-degenerate function over M . Then, the result follows from part (ii) of Corollary 4.4. **Q.E.D.**

Proposition 4.12 Let M be a non-empty compact set given by (4.13), $U \subset \mathbb{R}^n$ be an open set containing M and $f : U \mapsto \mathbb{R}$ be a twice continuously differentiable function. Assume that f is KKT-non-degenerate. If (f, M) satisfies (Q3), then (f, M) satisfies (P2).

Proof. Let $x \in M$ be a KKT point of f over M . Since f satisfies (Q3), x is a local minimum. Then by Corollary 4.3, $\text{KKTind}_f(x) = 1$, i.e. $\det(\text{KKT}_f(x)) > 0$. Hence, (f, M) satisfies (Q2) and the result follows from Proposition 4.11. **Q.E.D.**

We compare property (Q3) and Proposition 4.12 to the property (Q1) and Proposition 3.2, which are the main uniqueness results of this and the previous chapter ²⁰.

²⁰We note that Proposition 4.11 is indeed a stronger result, i.e. it asserts that (f, M) satisfies (P2) under weaker assumptions than in Proposition 4.12. However, we choose to do the comparison between Proposition 3.2 and Proposition 4.12, since assumptions (Q1) and (Q3) are more similar than assumptions (Q1) and (Q2).

Property (Q1) considers only the interior stationary points as candidates for minima and requires that every such point should be a strict local minimum. If one does not allow boundary stationary points to exist by means of a boundary condition, then Proposition 3.2 shows that this is sufficient to establish the uniqueness of the local minimum. In contrast, Property (Q3) treats the boundary points and the interior points uniformly, and requires that every stationary point (boundary or interior) should be a strict local minimum. We note that if (f, M) satisfies (Q1) and (B1), then it also satisfies (Q3), but not vice versa. Thus, in this aspect, Proposition 4.12 is a stronger uniqueness result than Proposition 3.2. We note, however, that Proposition 3.2 requires weaker structural assumptions for the function than Proposition 4.12 which requires the function to be twice continuously differentiable and non-degenerate.

Chapter 5

Conclusion and Future Research

In this thesis, we investigated the mountain pass theory of variational analysis and the index theory of differential topology to establish sufficient conditions for the uniqueness of the local optimum in a finite dimensional optimization problem. We proved results which establish uniqueness from local optimality conditions that can be checked algebraically. Our results are in contrast with the traditional approaches to the uniqueness problem which;

- (i) either establish uniqueness by sufficient convexity properties that are often too strict [cf. Section 2.1],
- (ii) or present theoretical characterizations which are equivalent to the uniqueness of the optimum but which cannot be checked algebraically [cf. Section 2.2].

We note that our results in index theory have implications beyond the uniqueness problem. Uniqueness is yet one application for the more general relationship we provide in Proposition 4.9 that is satisfied by the local properties of the function at its Karush-Kuhn-Tucker points. Similarly, Proposition 4.9 is yet one application of our generalized Poincare-Hopf theorem [cf. Theorem 4.1] which also has implications in fields such as equilibrium analysis and differential equations. The equilibrium of certain models could be characterized as the zero of a vector field (see, for example, the general economic equilibrium in Mas-Colell [34]). If we consider a general vector field instead of the gradient function, then Proposition 4.1 has implications on the existence and the uniqueness properties of equilibrium in such models. As one appli-

cation, Theorem 4.1 can be used to show the uniqueness of equilibrium under weaker assumptions than in Tang et al. [56], which investigates the equilibrium properties of a multi-protocol network congestion control model. Tang et al. [56] imposes strong boundary conditions on the model to be able to use the original Poincare-Hopf Theorem to establish the uniqueness of the equilibrium. Using Theorem 4.1 and treating the boundary and the interior equilibria uniformly, the uniqueness of the equilibrium can be established under weaker assumptions.

The following are some future research directions in our framework.

- We note that our uniqueness results from the constrained mountain pass theory and the index theory both require the region to satisfy a topological condition, albeit different conditions. In uniqueness results we obtain from the mountain pass theory, we require the region to be connected, whereas in uniqueness results we obtain from the index theory, we require the region to have Euler characteristic equal to 1. We do not yet fully understand the relationship between these conditions. How to reconcile these assumptions is a future research direction.
- Example 3.3 demonstrates that the (PS) condition required for establishing uniqueness in the unconstrained region case is not necessary. It would be a future research problem to investigate if the (PS) condition of Proposition 3.1 can further be relaxed.
- Proposition 3.3 can be proven using the methodology sketched.
- We believe that Theorem 4.1 can be further generalized. In particular, we think the requirement that the generalized critical points to be complementary could be relaxed (but it cannot be dropped altogether).

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